Nem Übbl 1 4.5.2007

A1 a z.z. Voraussetzungen des BFPS

- I abgeschlossen
- Abbbildungen in sich
- Kontraktionen

- $[0,1]$ abgeschlossen ✓
- $g(x) = \frac{1}{2}(x + e^{-x})$ g bildet $[0,1]$ in sich ab:

$g'(x) = \frac{1}{2}(1 - e^{-x}) \geq \ \forall x \in [0,1]$ d.h. $g(x)$ monoton wachsend $[0,1]$

somit $g(0) = \frac{1}{2} \leq g(x) \leq g(1) = \frac{1}{2}(1 + \frac{1}{e}) \approx 0{,}6839 < \underline{\underline{1}}$

gf g ist auf $[0,1]$ eine Kontraktion

g stetig diffbar, zu zeigen: $|g'(x)| \leq L < 1 \ \forall x \in [0,1]$

$g''(x) = \frac{1}{2}e^{-x} \geq 0$ d.h. $g'(x)$ monoton wachsend

Somit $g'(0) = 0 \leq g'(x) \leq g'(1) = \frac{1}{2}(1 - \frac{1}{e}) \approx 0{,}3161 < 1$ $\forall x \in [0,1]$

$h(y) = \frac{1}{6}(5y + e^{-y})$ h bildet $[0,1]$ in sich ab

$h'(y) = \frac{1}{6}(5 - e^{-y}) \geq 0$ h monoton wachsend $\forall y \in [0,1]$

Somit $h(0) = \frac{1}{6} \leq hy \leq h(1) = \frac{1}{6}(5 + \frac{1}{e}) \approx 0{,}8946 < 1$

. h ist auf $[0,1]$ kontrahierend

$h''(y) = \frac{1}{6}e^{-y} \geq 0 \rightarrow h'$ monoton wachsend

somit $h'(0) = \frac{2}{3} \leq h'(y) \leq h'(1) = \frac{1}{6}(5 - \frac{1}{e}) \approx 0{,}7720 < 1 \ \forall y \in [0,1]$

b) $x^* = y^*$ Es gilt $x^* = \frac{1}{2}(x^* + e^{-x^*}) \Leftrightarrow x^* = e^{-x^*}$

$y^* = \frac{1}{6}(x^* + e^{-y^*}) \Leftrightarrow y^* = e^{-y^*}$

d.h. x^* und y^* sind Nullstellen von $f(t) = t - e^{-t}$

$f'(t) = 1 + e^{-t} > 0 \Rightarrow f$ ist monoton wachsend

$\Rightarrow f$ hat höchstens eine Nullstelle

$|x_n - x^*| \leq \frac{L_g^n}{1 - L_g}|x_0 - x_1| = \frac{0{,}3161^n}{0{,}6839}|0 - \frac{1}{2}| = \cancel{0{,}3\ldots} \frac{0{,}3161^n}{1{,}3678} < 10^{-3}$!

$\Rightarrow n \cdot \log(0{,}3161) < \log(1{,}3678) - 3$

$\Rightarrow n > 5{,}7259 \Rightarrow n = 6$

$$|y_n - y^*| \le \frac{L^n}{1-L_n} |y_0 - y_1| = \frac{0{,}772^n}{0{,}228} \left|0 - \frac{1}{8}\right|$$

$$= \frac{0{,}772^n}{1{,}368} \overset{!}{<} 10^{-3}$$

$$\Rightarrow n \log(0{,}772) < \log(1{,}368) - 3$$

$$\Rightarrow n > 25{,}483\cancel{8}$$

$$\underline{\underline{n = 26}}$$

$\underline{2}$ i) $f(x) = 7 - \frac{1}{5}\cos 3x$, $[a,b] = [0, \frac{2}{3}\pi]$,

$f(x) = C^1([a,b])$, $f'(x) = \frac{3}{5}\sin 3x$

$|f'(x)| \le f'(\frac{\pi}{6}) = \frac{3}{5} \cdot \sin\frac{\pi}{2} = \frac{3}{5}$

$\overset{MWS}{\Rightarrow}$ $L = \frac{3}{5}$

ii) $f(x) = 3 + \frac{1}{3}|x|$; $[a,b] = [-1,1]$

$|f(x) - f(y)| = |3 + \frac{1}{3}|x| - 3 - \frac{1}{3}|y|| = \frac{1}{3}||x|-|y|| \le \frac{1}{3}|x-y|$

$L = \frac{1}{3}$ für $x = 1, y = 0$ gleichheit

iii) $f(x) = x + \frac{1}{x}$, $[a,b] = [2,4]$

$f \in C^1[a,b]$, $f'(x) = 1 - \frac{1}{x^2}$

$|f'(x)| \le f'(x) = 1 - \frac{1}{16} = \frac{15}{16}$

$\underline{MWS} \Rightarrow L = \frac{15}{16}$

b) $f(x) = \sqrt{x}$ $f: [0,1] \to \mathbb{R}$

Ann. $\exists L > 0$ mit $|f(x) - f(y)| \le L|x-y|$

Sei $x_n = \frac{1}{n}$, $y_n = \frac{1}{4n}$ ($n \in \mathbb{N}$ beliebig)

$|f(x_n) - f(y_n)| \le L|x_n - y_n|$

$\Leftrightarrow \left|\frac{1}{\sqrt{n}} - \frac{1}{2\sqrt{n}}\right| \le L \cdot \frac{3}{4} \cdot \frac{1}{n} \Leftrightarrow \frac{1}{2}\frac{1}{\sqrt{n}} \le L\frac{3}{4}\cdot\frac{1}{n}$

$\Rightarrow L \ge \frac{2}{3}\sqrt{n}$

$L \ge \frac{2}{3}\sqrt{n}$ $\forall n \in \mathbb{N}$ $\lightning$

weil $\frac{2}{3}\sqrt{n} \xrightarrow[n\to\infty]{} \infty$

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[3] Vor BFPS - I abgeschlossen
- Abbildung in sich
- Kontraktion

a) $I = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ abgeschlossen ✓

- $\frac{1}{2} - \frac{1}{40}x^2 - \frac{1}{50}y^2$ bzw $\frac{1}{2} - \frac{1}{40}x^2 - \frac{1}{20}y^2$

sind in x und y monoton fallend

$0 \le \frac{1}{2} - \frac{1}{40}\left(\frac{1}{2}\right)^2 - \frac{1}{50}\left(\frac{1}{2}\right)^2 = \frac{391}{800} \le \frac{1}{2} - \frac{1}{40}x^2 - \frac{1}{50}y^2 \le \frac{1}{2}$

$0 \le \frac{1}{2} - \frac{1}{40}\left(\frac{1}{2}\right)^2 - \frac{1}{20}\left(\frac{1}{2}\right)^2 \frac{77}{160} \le \frac{1}{2} - \frac{1}{40}x^2 - \frac{1}{20}y^2 \le \frac{1}{2}$

$\left\| F\binom{x}{y} - F\binom{\tilde{x}}{\tilde{y}} \right\|_\infty = \max \left\{ \left| \frac{1}{40}(x^2 - \tilde{x}^2) + \frac{1}{50}(y^2 - \tilde{y}^2) \right|, \right.$

$\left. \left| \frac{1}{40}(x^2 - \tilde{x}^2) + \frac{1}{20}(y^2 - \tilde{y}^2) \right| \right\} \le$

$\le \max \left\{ \left| \frac{1}{40}(x + \tilde{x}) + \frac{1}{50}(y + \tilde{y}) \right|, \left| \frac{1}{40}(x + \tilde{x}) + \frac{1}{20}(y + \tilde{y}) \right| \right\} \cdot$

$\cdot \max \{ |x - \tilde{x}|, |y - \tilde{y}| \}$

$\le \max \left\{ \frac{1}{40} + \frac{1}{50}, \frac{1}{40} + \frac{1}{20} \right\} \cdot \max \{ |x - \tilde{x}|, |y - \tilde{y}| \}$

$= \cancel{\max} \frac{3}{40} \left\| \binom{x}{y} - \binom{\tilde{x}}{\tilde{y}} \right\|_\infty \quad L = \frac{3}{40}$ ✓

b) $\binom{x_{n+1}}{y_{n+1}} = F \binom{x_n}{y_n}$

n	x_n	y_n	$\tilde{x}_n$	$\tilde{y}$
0	$1/2$	$1/2$	0,5	0,5
1	$\frac{391}{800}$	$\frac{77}{160}$	0,48875	0,48125
~~2~~				
2	$\frac{125\,528\,531}{256\,000\,000}$	$\frac{12350669}{25600000}$	0,489396	0,48248

$x^* \approx 0,489355$

$y^* \approx 0,482375$

α) $\left\| \binom{x_2}{y_2} - \binom{x^*}{y^*} \right\| \le \frac{L}{1-L} \left\| \binom{x_2}{y_2} - \binom{x^*}{y^*} \right\| \approx \frac{0,075}{0,925} \cdot 1.2 \cdot 10^{-3}$

$< 10^{-4}$

β) $\left\| \binom{x_n}{y_n} - \binom{x^*}{y^*} \right\| \le \frac{L^n}{1-L} \left\| \binom{x_0}{y_0} - \binom{x^*}{y^*} \right\| =$

$= \frac{0,075^n}{0,925} \cdot 0,1875 \cdot 10^{-1} \overset{!}{<} 10^{-8}$

$\Rightarrow 0,075^n \le 10^{-7} \frac{0,925}{0,1875} \quad n \log(0,075) < -7 + \log\left(\frac{0,925}{0,1875}\right)$

$\Rightarrow n \ge 5,6$ also $n = 6$

$$G\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}x\\y\end{pmatrix} - F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} x - \frac{1}{2} + \frac{1}{40}x^2 + \frac{1}{50}y^2 \\ y - \frac{1}{2} + \frac{1}{40}x^2 + \frac{1}{20}y^2 \end{pmatrix}$$

$$\begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix} = \begin{pmatrix}x_n\\y_n\end{pmatrix} - J_G^{-1}\begin{pmatrix}x_n\\y_n\end{pmatrix} G\begin{pmatrix}x_n\\y_n\end{pmatrix} \quad n \geq 0$$

Praxis Lsg: $J_G\begin{pmatrix}x_n\\y_n\end{pmatrix}\left[\begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix} - \begin{pmatrix}x_n\\y_n\end{pmatrix}\right] = -G\begin{pmatrix}x_n\\y_n\end{pmatrix} \quad \begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix} - \begin{pmatrix}x_n\\y_n\end{pmatrix} =: \begin{pmatrix}p_{n+1}\\q_{n+1}\end{pmatrix}$

$$J_G\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{20}x & \frac{1}{25}y \\ \frac{1}{20}x & 1 + \frac{1}{10}y \end{pmatrix}$$ Hier $n = 0,\ x_0 = y_0 = \frac{1}{2}$

$$\begin{pmatrix} \frac{41}{40} & \frac{1}{50} \\ \frac{1}{40} & \frac{21}{20} \end{pmatrix}\begin{pmatrix}p_1\\q_1\end{pmatrix} = -\begin{pmatrix}\frac{9}{800}\\ \frac{3}{160}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}p_1\\q_1\end{pmatrix} \approx -\begin{pmatrix}0{,}01063211721\\0{,}0176039972\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x_1\\y_1\end{pmatrix} \approx \begin{pmatrix}0{,}4893678829\\0{,}4823960028\end{pmatrix}$$

(4) a) $\Leftrightarrow$

$\Rightarrow$ für $x = y$ ✓

sei f konvex $x, y \in [a,b]$ $x \neq y$ $\alpha \in (0,1)$

$$\frac{1}{1-\alpha} f(\alpha x + (1-\alpha)y) \leq \frac{\alpha}{1-\alpha} f(x) + f(y) = -f(x) + \frac{1}{1-\alpha} f(x) + f(y)$$

$$\frac{1}{1-\alpha} f(-(1-\alpha)x + x + (1-\alpha)y) - \frac{1}{1-\alpha} f(x) \leq f(y) - f(x)$$

$$\frac{1}{1-\alpha}\left(f(x + (1-\alpha)(y-x)) - f(x)\right) \leq f(y) - f(x) \qquad \beta = 1-\alpha$$

$$\frac{1}{\beta}\left(f(x + \beta(y-x)) - f(x)\right) = \underbrace{\frac{f(x+\beta(y-x)) - f(x)}{\beta(y-x)}}_{\xrightarrow{\beta \to 0} f'(x)}(y-x) \leq f(y) - f(x)$$

($\Leftarrow$) $x, y \in [a,b]$ $\alpha \in (0,1)$ bel. fest

$z := \alpha x + (1-\alpha)y \in [a,b]$

gilt (i) $f(x) - f(z) \geq f'(z)(x-z)$

(ii) $f(y) - f(z) \geq f'(z)(y-z)$

α (i) $+ (1-\alpha) \cdot$ (ii)

$$\alpha f(x) + (1-\alpha) f(y) - f(z) \geq f'(z)(\alpha x + (1-\alpha)y - z) = 0$$

$$\Leftrightarrow \alpha f(x) + (1-\alpha) f(y) \geq f(z) = f(\alpha x + (1-\alpha)y) \rightarrow \text{konvex}$$

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! Interned → ~~B~~ Dort wo es Gbbl. gibt

A5 b) $\|Ax\|_2^2 = \sum_{i=1}^{n} (Ax)^2 = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk} x_k \right)^2 \underset{CSU}{\le}$

$\sum \sum a_{jk}^2 \; \sum x_\ell^2 = N_F(A)^2 \cdot \|x\|_2^2$

c) $\|Ax\|_\infty = \max_{1\le j\le n} |(Ax)_j| = \max_{1\le j\le n} \left| \sum_{k=1}^{n} a_{jk} x \right| \underset{\Delta}{\le} \max_{1\le j\le n} \sum_{k=1}^{n} |a_{jk}| |x_\ell| \le$

$\le \max_{1\le j\le n} \sum_{k}^{n} |a_{jk}| \max_{1\le \ell\le n} |x_\ell| = N_7(A) \cdot \|x\|_\infty$

! d) Submultiplikativität von $N(\cdot)$

$N(A^\ell) \le N(A^{\ell-1}) \cdot N(A) \le (N(A))^\ell$

$0 \le (N(A))^\ell \le \underbrace{(N(A))}_{<1}{}^\ell \xrightarrow[\ell\to\infty]{} 0$

Definitheit $\Rightarrow \lim_{\ell\to\infty} A^\ell = 0$

Umkehrung gilt nicht

$A := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow A^\ell = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \ell = 2 \quad \lim_{\ell\to\infty} A^\ell = 0$ Aber $N_7(A) = 2$

$\boxed{A6}$ GSV konvergiert für ein LGS mit diagonaldominanter Matrix

Zeilensummenkriterium: Für $\alpha \in (-3, 3)$

$$\left.\begin{aligned} \sum_{j=2}^{3} |a_{1j}| &= 1 < 2 = |a_{11}| \\ \sum_{j=1,3} |a_{2j}| &= 1 + |\alpha| < 4 = |a_{22}| \\ \sum_{j=1}^{2} |a_{3j}| &= 1 < 4 = |a_{33}| \end{aligned}\right\}$$

Diagonaldominanz
⇒ Konvergenz
⇒ GSV $\alpha \in (-3, 3)$

b) $T_{ESV} = -(D+L)^{-1} U \qquad A = \begin{pmatrix} L & D & U \end{pmatrix}^n$

$D + L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix} \qquad (D+L)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & 0 \\ -\frac{1}{8} & 0 & \frac{1}{4} \end{pmatrix}$

$-(D+L)^{-1} \cdot U = -\begin{pmatrix} +\frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & 0 \\ -\frac{1}{8} & 0 & \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & +\frac{1}{2} \\ 0 & 0 & -\frac{1}{8} - \frac{\alpha}{4} \\ 0 & 0 & -\frac{1}{8} \end{pmatrix} =$

$= T_{ESV} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{8} \\ 0 & 0 & -\frac{1}{8} \end{pmatrix} \qquad g = (L+D)^{-1} b =$

$= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & 0 \\ -\frac{1}{8} & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$$x_1 = T_{ESV}\, x_0 + g = g = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$x_2 = T_{ESV}\, x_1 + g = g = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$x_3 = x_2 \Rightarrow (= x_4 = x_i \mid i \in \mathbb{N} \}$$

c) $$GSV: T_{GSV} = -D^{-1}(L+U) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ -\frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & 0 \end{pmatrix}$$

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$\boxed{A10, A9}$ sieh HP.

$\boxed{A11}$ $p(x) = 16x^4 - 12x^2 + 1$

Gerschgorin: Die NST $\xi_1, \dots, \xi_n$ des Polynoms

$p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ liegen in einem Kreis um 0

mit Radius $r \le \max\{1, \sum_{i=0}^{n-1} a_i\}$;

$\le \max\{|a_0|, 1+|a_1|, \dots, 1+|a_{n-1}|\}$

Normalisiere $p(x)$

$\tilde{p}(x) = x^4 - \frac{12}{16}x^2 + \frac{1}{16} = x^4 - \frac{3}{4}x^2 + \frac{1}{16}$

$r \le \max\{1, \frac{3}{4} + \frac{1}{16}\} = 1$

$r \le \max\{\frac{1}{16}, 1 + \frac{3}{4}\} = \frac{7}{4}$ Kreis um 0 ~~und bis 1~~ mit $r = 1$

alle NST von p liegen in $K(0, 1)$

b) Konstruktion einer Sturmschen Kette

~~p_0~~ $p_0(x) = p(x) = 16x^4 - 12x^2 + 1$

$p_1(x) = p'(x) = 64x^3 - 24x$

$p_1(x) \cdot \frac{x}{4} = 16x^4 - 6x^2 \Rightarrow p_0 = p_1 \cdot \frac{x}{4} \underbrace{(6x^2 - 1)}_{= p_2(x)}$

$\Rightarrow p_2(x) = 6x^2 - 1 \Rightarrow p_2(x) \cdot \frac{32}{3}x = 64x^3 - \frac{32}{3}x \Rightarrow p_1(x) = p_2(x)\frac{32}{3}x - (\frac{40}{3}x)$

$\Rightarrow p_3(x) = \cancel{\frac{9}{20}x}\ \frac{40}{3}x$

$p_3(x) \cdot \frac{9}{200}x = 6x^2 \Rightarrow p_2(x) = p_3(x) \cdot \frac{9}{20}x - (1)$

$p_4(x) = 1$

p_0, p_1, p_2, p_3, p_4 bilden Sturmsche Kette

Untersuche VZW für $x = \pm 1, \pm \frac{1}{2}, 0$

x	p_0	p_1	p_2	p_3	p_4	VZW
-1	5	-40	5	$-\frac{40}{3}$	1	4
$-\frac{1}{2}$	-1	4	$1/2$	$-\frac{20}{3}$	1	3
0	1	0	-1	0	1	2
$1/2$	-1	-4	$1/2$	$\frac{20}{3}$	1	1
1	5	40	5	$\frac{40}{3}$	1	0

$Z_{-1}^{-\frac{1}{2}} = 4 - 3 = 1$

$Z_{-\frac{1}{2}}^{0} = 3 - 2 = 1$

$Z_0^{1/2} = 1$

$Z_{1/2}^{1} = 1$

Satz von Sturm: $Z_\alpha^\beta(p_0) = W(p_0(\alpha), \dots, p_r(\alpha))$ $- W(p_0(\beta), \dots p_r(\beta))$

c) Newtonverfahren $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{16x_k^4 - 12x_k^2 + 1}{64x_k^3 - 24x_k}$

$$x_{k+1} = \frac{48x_k^4 - 12x_k^2 - 1}{64x_k^3 - 24x_k}$$

$x_0 = 1, \quad x_1 = \frac{7}{8} \approx 0{,}875, \quad x_2 = \frac{919}{1120} \approx 0{,}8205 \quad x_3 = 0{,}895$

$x_6 \approx x_7 \approx 0{,}809016914\,37495$

A12 a) $p(x) = 2x^5 - 3x^4 - 20x^2 - 10x - 40$

$p^{(\nu)}(3)$, $\nu = 0, .., 5$, Umentwicklung $(x-3)$

vollst HS

	2	-3	0	-20	-10	-40	
$x=3$	-	6	9	27	21	33	
	2	3	9	7	11	$\boxed{-7}$	$= p(3$
$x=3$	-	6	27	108	345		
	2	9	36	115	$\boxed{356}$		$= p'(3)$
$x=3$	-	6	45	243			
	2	15	81	$\boxed{318}$			$= p''(3) \cdot \frac{1}{2!}$
$x=3$	-	6	63				
	2	21	$\boxed{144}$				$= p'''(3) \cdot \frac{1}{3!}$
	-	6					
	$\boxed{2}$	$\boxed{27}$					$= p^{(4)}(3) \cdot \frac{1}{4!}$

$p^{(5)}(3) \cdot \frac{1}{5!}$

$\Rightarrow p'(3) =$

$p(x) = p(3) + p'(3)(x-3) + \frac{1}{2!} p''(3)(x-3)^2 + \dots + \frac{1}{5!} p^{(5)}(3)(x-3)^5$

$p(x) = -7 +$

b) z.z. $\int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases}$ $\quad T_k(x) = \cos(k \arccos x)$ $k \geq 0$

1. Fall $(m \neq n)$ $\int_{-1}^{1} \cos(n \arccos x) \cos(m \arccos x) \frac{dx}{\sqrt{1-x^2}} = \int_{\pi}^{0} \cos(ny)\cos(my) \frac{-\sin y \, dy}{\sqrt{1-\cos^2 y}}$

$\left\{ \begin{array}{l} x = \cos y \\ dx = -\sin y \, dy \end{array} \right\}$

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$$\int_0^\pi \cos(ny)\cos(my)\,dy \qquad \left.\begin{array}{l}\cos(x\pm y)\\ = \cos x\cos y\\ \mp \sin x \sin y\end{array}\right.$$

$$= \frac{1}{2}\left[\int_0^\pi \cos(n+m)y + \cos(n-m)y\,dy\right] = \frac{1}{2}\left[\frac{1}{n+m}\sin(n+m)y + \frac{1}{n-m}\sin(n-m)y\right]_0^\pi = 0$$

2. Fall $n = m \neq 0$

$$\int_{-1}^{1} \cos(n\arccos(x))^2 \frac{dx}{\sqrt{1-x^2}} \underset{\substack{x=\cos y\\ dx=-\sin y\,dy}}{=} \int_0^\pi \underbrace{\cos^2(ny)}_{\cos 2x = \cos^2 x - \sin^2 x}\,dy =$$

$$= \left[\frac{1}{2}y + \frac{1}{4y}\sin 2ny\right]_0^\pi = \frac{\pi}{2}$$

3. Fall $n = m = 0$ $\quad T_0(x) = 1$

$$\int_{-1}^{1} \cancel{\cos(} \frac{1}{\sqrt{1-x^2}}\,dx \underset{\substack{x=\cos y\\ dx=-\sin y\,dy}}{=} \int_0^\pi 1\,dy = \pi$$

e)

$$S(x) = \begin{cases} S_1(x) = \frac{1}{6}(x^2+6x^2+12x+8) & -2 \le x - 1 \\ S_2(x) = \frac{1}{6}(x^3+6x^2+12x+8) & -1 \le x < 0 \\ S_3(x) = & 0 \le x < 1 \\ S_4(x) & 1 \le x \le 2 \end{cases}$$

$\Delta = \{x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 2\}$ kub. Spline

Dh. (1) $S|_{[x_{j-1}, x_j]} \in P_j \quad j = 1, \dots 4$ $\quad\Big|\quad S|_{[x_{j-1}, x_j]} = S_1 \quad j = 1, \dots, 4$

(2) $S \in C^2[x_0, x_4]$ $\quad\Big|\quad \Rightarrow S|_{[x_{j-1}, x_j]} \in P_3,\ S \in C^2(x_{j-1}, x_j)$

$j = 1, \dots, 4$

Unters. der Ansatzstellen

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A15 Für $\varepsilon \neq -\frac{7}{2}$ $\quad A^{-1} = \frac{1}{7+2\varepsilon} \begin{pmatrix} 4+\varepsilon & 1 \\ 1 & 2 \end{pmatrix}$

$$\mathrm{cond}_2(A) = N_7(A) \cdot N_7(A^{-1}) = \frac{1}{7+2\varepsilon} \max\{3, 1+|4+\varepsilon|\}^2$$

Schlechte Kondition: cond(A) groß, dh für $\varepsilon \to -7/2$ und $|\varepsilon| \to \infty$

b) $\hat{A} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/_{4+\varepsilon} \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 4+\varepsilon \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{4+\varepsilon} & 1 \end{pmatrix}$ $\quad \varepsilon \neq -4$

$\hat{A}^{-1} = \frac{8-2\varepsilon}{7+2\varepsilon} \begin{pmatrix} 1 & -1/2 \\ 1/_{4+\varepsilon} & 1 \end{pmatrix}$ $\quad \varepsilon \neq -\frac{7}{2}$ $\quad \mathrm{cond}_2(\hat{A}) = \left|\frac{8+2\varepsilon}{7+2\varepsilon}\right| \max\left\{\frac{3}{2}, 1+\frac{1}{|4+\varepsilon|}\right\}^2$

$\varepsilon \to \frac{7}{2}$: $\mathrm{cond}_2(A) \to \infty$, $|\varepsilon| \to \infty$, Keine schlechte Kondition $\left|\frac{8+2\varepsilon}{7+2\varepsilon}\right| \to 1$

$\varepsilon \to (-4)$: $(\varepsilon > -4)$ $\quad \max\left\{\frac{3}{2}, 1+\frac{1}{|4+\varepsilon|}\right\}^2 = \left\{1+\frac{1}{4+\varepsilon}\right\}^2 = \left(\frac{5+\varepsilon}{4+\varepsilon}\right)^2$

$\Rightarrow \mathrm{cond}_2 A = -\frac{8+2\varepsilon}{7+2\varepsilon} \frac{(5+\varepsilon)^2}{(4+\varepsilon)^2} \overset{x=4+\varepsilon}{=} -\frac{2x^2+4x+2}{2x^2-x} \xrightarrow[x\to\infty]{} \infty$

c)

ε	$\mathrm{cond}_2(A)$		$\mathrm{cond}_2(\hat{A})$
0	$\frac{25}{7}$	>	$\frac{18}{7}$
-1	$\frac{16}{5}$	>	$\frac{27}{20}$
$-\frac{15}{4}$	18	<	25

A16 a) Doppelzeiliges Hornerschema

	x^4	x^3	x^2	x	1
	1	-2	-6	22	-15
$s=4$	–	4	8	-12	–
$t=-5$	–	–	-5	-10	15
	1	2	-3	$\boxed{0}$	$\boxed{0}$

$r(x) = x^2+2x-3$

$= (x+a)(x+b)$

$x = 1$ NS

$\Rightarrow$

$p(x) = x^4 - 2x^3 - 6x^2 + 22 - 15$

$q(x) = x^2 - 4x + 5 \qquad q(x) = x^2 - sx - t$

$r(x) = p(x) / q(x)$

$r(x)$:

	1	+2	-3
$a=1$	–	1	3
	1	3	0

$r(x) = (x-1)(x+3)$

c) $p''(1)$, $p'''(-2)$

	1	-2	-6	22	-15
$x=1$	–	1	-1	-7	15
	1	-1	-7	15	$\boxed{0}$
$x=1$	–	1	0	-7	
	1	0	-7	$\boxed{8}$	$= p'(1)$
$x=1$	–	1	1		
	1	1	$\boxed{-6}$	$= p''(1) \cdot \frac{1}{2}$	$\Rightarrow p''(1) = -12$

$p'''(-2)$

	1	-2	-6	22	-15	
$x=-2$	-	-2	8	-4	-36	
	1	-4	2	18	$\boxed{-51}$	$= p(-2)$
$x=-2$	-	-2	12	-28		
	1	-6	14	$\boxed{-10}$		$= p'(-2)$
$x=-2$	-	-2	16			
	1	-8	$\boxed{30}$			$= \frac{1}{2} p''(-2)$
$x=-2$	-	-2	$\cancel{20}$			
	1	$\boxed{-10}$				$= \frac{1}{6} p'''(-2)$

$$= \frac{1}{6} p'''(-2) \Rightarrow \boxed{p'''(\cancel{4}-2) = -60}$$

$\boxed{A17}$ $T_n(x) = \cos(n \arccos x)$ $n \geq 0$

z.z. $T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$, $n \geq 1$, $T_0(x) = 1$, $T_1(x) = x$

Bew: $T_0(x) = \cos(0 \cdot \arccos x) = 1$ $T_1(x) = \cos(\arccos x) = x$

Add Theorem: $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$n \geq 1$ $T_{n+1}(x) = \cos((n+1)\arccos x) = \cos(n \arccos x)\cos(\arccos x) -$
$- \sin(n \arccos x)\sin(\arccos x) =$

$= T_n(x) \cdot x - (\cos((n-1)\arccos x) - (\cos(n \arccos x)\cos(\arccos x)))$

$= T_n(x)\, x - (T_{n-1}(x) - T_n(x) \cdot x) = 2x\, T_n(x) - T_{n-1}(x)$ □

b) $T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x\, x - 1 = 2x^2 - 1$

$T_3(x) = 2x(2x^2-1) - x = 4x^3 - 3x$

$T_4(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 3x^2 + 1$

$T_5(x) = 2x(8x^4 - 3x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x$

$T_6(x) = 2x(16x^5 - 10x^3 + 5x) - (8x^4 - 3x^2 + 1) = 32x^6 - 48x^4 + 18x^2 - 1$

$p(x) = 16x^6 - 8x^5 - 16x^4 + 12x^2 + x + 3$

$$p(x) = \sum_{j=0}^{6} a_j\, T_j(x)$$

x^6 : $a_6 = \frac{1}{2}$

x^5 : $a_5 = -\frac{1}{2}$

x^4 : $a_4 = \frac{-16+24}{8} = 1$

x^3 : $a_3 = \frac{12-10}{4} = 1/2$

x^2 : $a_2 = \frac{0-9+8}{2} = -1/2$

x : $a_1 = \frac{1 + \frac{5}{2} + 3/2}{1} = 5$

x^0 : $a_0 = \frac{3 + 1/2 - 1 - \frac{1}{2}}{1} = 2$

$p(x) = \frac{1}{2} T_6 - \frac{1}{2} T_5 + T_4 + \frac{1}{2} T_3 - \frac{1}{2} T_2 + 5\, T_1 + 2\, T_0$

Numerik / 13. 7. 2001

$\boxed{A23}$ geg: pos. Gewfktn $\omega(x): [-1,1] \to \mathbb{R}$, GQF $-1 \le x_1 < x_2 < \dots < x_s = 1$

Dh. $\int_{-1}^{1} \omega(x) f(x)\,dx = \sum_{j=1}^{s} w_j f(x_j) + Rf$; Knoten x_j sind Nullstellen

des Orthogonalpolynoms p_s bzgl $\langle f,g \rangle_\omega := \int_{-1}^{1} \omega(x) f(x) g(x)\,dx$

$w_j = \int_{-1}^{1} \omega(x)\, \ell_j(x)\,dx$ (d.h. interpolatorisch)

z.z. GQF ist exakt für Polynome $p \le \mathcal{P}_{2s-1}$

<u>1. Schritt</u> GQF exakt für $p \in \mathcal{P}_{s-1}$

Sei $p(x) \in \mathcal{P}_{s-1}$, dann gilt $p(x) = \sum_{j=1}^{s} p(x_j)\, \ell_j(x)dx$

$$\int_{-1}^{1} \omega(x) p(x)\,dx = \int_{-1}^{1} \omega(x) \sum_{j=1}^{s} p(x_j)\, \ell_j(x)\,dx = \sum_{j=1}^{s} p(x_j) \int_{-1}^{1} \omega(x)\, \ell_j(x)\,dx = \sum_{j=1}^{s} p(x)\, \omega_j$$

<u>2. Schritt</u> $p(x) \in \mathcal{P}_{2s-1}$ Euklidescher Algor.

$\Rightarrow \exists\, r(x_j)\, s(x) \in \mathcal{P}_{s-1}$ mit $p(x) = s(x)\, p_s(x) + r(x)$

$$\int_{-1}^{1} \omega(x) p(x)\,dx = \underbrace{\int_{-1}^{1} \omega(x)\, s(x)\, p_s(x)\,dx}_{=0} + \int_{-1}^{1} \omega(x)\, r(x)\,dx \underset{r \in \mathcal{P}_{s-1}}{=} \sum_{j=1}^{s} w_j\, r(x_j) =$$

$$\sum_{j=1}^{s} w_j \big(s(x_j) \underbrace{p_s(x_j)}_{=0} + r(x_j) \big) = \sum_{j=1}^{s} w_j\, p(x_j) \qquad \square$$

b) Gewichte w_j sind positiv

$q(x) = \left(\frac{p_s(x)}{x - x_i} \right)^2 \in \mathcal{P}_{2s-1}$ $\quad i \in \{1, \dots, s\}$ bel. fest

Es gilt $q(x_j) = 0$, $j \ne i$, $q(x_i) = (p_s'(x_i))^2$

$$p_s(x) = \prod_{j=1}^{s} (x - x_j), \quad p_s'(x) = \prod_{\substack{j=1\\ j\ne i}}^{s} (x - x_j) + (x - x_i) \Big(\prod_{\substack{j=1\\ j \ne i}} (x - x_j) \Big)'$$

$$p_s'(x_i)^2 = \Big(\prod_{\substack{j=1\\ j\ne i}}^{s} (x_i - x_j) \Big)^2 = q(x_i)$$

Damit $\int_{-1}^{1} \omega(x)\, q(x)\,dx = \sum_{j=1}^{s} w_j\, q(x_j) = w_i\, q(x_i) = w_i\, (p_s'(x_i))^2$

$$w_i = \frac{1}{(p_s'(x_i))^2} \int_{-1}^{1} \omega(x)\, q(x)\,dx = \int_{-1}^{1} \omega(x) \left(\frac{p_s(x)}{(x - x_i)\, p_s'(x_i)} \right)^2 dx > 0$$

$\boxed{A24}$ $x_1 = -1$, $x_{s+1} = 1$, $x_2, \dots, x_s$ noch zu wählen.

Sei $p \in \mathcal{P}_{s-2}$, dann soll gelten: $\int_{-1}^{1} \underbrace{(x - x_2) \cdots (x - x_s)(1 - x^2)\, p(x)}_{= q(x) \in \mathcal{P}_{2s-1}}\,dx = \sum_{k=1}^{s+1} w_k\, q(x_k) = 0$

Damit gilt $\int_{-1}^{1} (x - x_2) \cdots (x - x_s) \underbrace{(1 - x^2)}_{\ge 0}\,dx = 0 \quad \forall p \in \mathcal{P}_{s-2}$

$\Rightarrow (x - x_2) \cdots (x - x_s)$ ist Orthogonalpolynom

vom Grad $s-1$ bzgl $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, (1 - x^2)\,dx$ $\quad \begin{pmatrix} \omega = (1 - x^2) \\ [-1, 1] \end{pmatrix}$

$\Rightarrow x_2, \dots, x_5$ sind die Nullstellen dieses Orthogonalpolynoms $\left(\begin{matrix} w = (1-x^2) \\ [-1,1] \end{matrix}\right)$

b) Es gilt: $\int_{-1}^{1} p(x)\,dx = \omega_1 p(-1) + \omega_2 p(x_2) + \omega_3 (x_3) + \omega_4 (\cancel{x_4}\, 1) \quad \forall p \in \mathcal{P}_5$

$(1-x^2)$, $[-1,1]$, x_2, x_3 symmetrisch zu $x=0$

$\omega_1 = \omega_4$, $\omega_2 = \omega_3$, $x_2 = -x_3$

$\int_{-1}^{1} p(x)\,dx = \omega_1 (p(-1) + p(1)) + \omega_2 (p(x_2) + p(-x_2)) \quad \forall p \in \mathcal{P}_5$

Betrachte gerade Polynome vom Grad 0, 2, 4

$p = 1 \Rightarrow 2 = 2(\omega_1 + \omega_2) \Leftrightarrow \omega_1 + \omega_2 = 1 \quad (1)$

$p = 1 - x^2 \Rightarrow \frac{4}{3} = 2\omega_2 (1 - x_2^2) \quad (2)$

$p = 1 - x^4 \Rightarrow \frac{8}{5} = 2\omega_2 (1 - x_2^4) \quad (3)$

$(3)/(2) \Rightarrow \frac{6}{5} = 1 + x_2^2 \Rightarrow x_2 = -\sqrt{\frac{1}{5}} \dots, \quad x_3 = \sqrt{\frac{1}{5}}$

$(1), (2) \Rightarrow \omega_2 = \omega_3 = \frac{5}{6}$ und $\omega_1 = \omega_4 = \frac{1}{6}$

QF heißt Gauß-Lobatto QF

$\boxed{A25}$ Trigonometrische IP: $x_j = \frac{2\pi j}{2m+1} \quad j = 0, \dots, 2m$

$\tau_m(x) = \frac{\alpha_0}{2} + \sum_{\nu=1}^{m} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x)$

$\left.\begin{matrix}\alpha_\nu \\ \beta_\nu\end{matrix}\right\} = \frac{2}{2m+1} \sum_{j=0}^{2m} f(x_j) \left\{\begin{matrix} \cos \nu x_j & \nu = 0, \dots, m \\ \sin \nu x_j & \nu = 1, \dots, m \end{matrix}\right.$

a) $\alpha_0 = \frac{2}{3} \sum_{j=0}^{2} f(x_j) \underbrace{\cos(0 \cdot x_j)}_{=1} = \frac{2}{3}(2 - 1 + 4) = \frac{10}{3}$

$\alpha_1 = \frac{2}{3} \sum_{j=0}^{2} f(x_j) \cos(x_j) = \frac{2}{3}\left(2 \cdot 1 - 1\left(-\frac{1}{2}\right) + 4\left(-\frac{1}{2}\right)\right) = \frac{1}{3}$

$\beta_1 = \frac{2}{3} \sum_{j=0}^{2} f(x_j) \sin(x_j) = \frac{2}{3}\left(2 \cdot 0 - 1\left(\frac{1}{2}\sqrt{3}\right) + 4\left(-\frac{1}{2}\sqrt{3}\right)\right) = -\frac{5}{3}\sqrt{3}$

$\tau_1(x) = \frac{5}{3} + \frac{1}{3}\cos x - \frac{5}{3}\sqrt{3} \sin x$

c) $\tau_1\left(\frac{9}{4}\pi\right) = \tau_1\left(\frac{\pi}{4}\right) = \frac{5}{3} + \frac{1}{3}\frac{1}{2}\sqrt{2} - \frac{5}{3}\sqrt{3}\frac{1}{2}\sqrt{2} = \frac{5}{3} + \frac{\sqrt{2}}{6} - \frac{5}{6}\sqrt{6} \approx -0.13887$

$\boxed{A26}$ $T_3(x) = 4x^3 - 3x \Rightarrow x_0 = 0, \; x_{1/2} = \pm \frac{1}{2}\sqrt{3}$

Bew: $\int_{-1}^{1} \frac{p(x)}{\sqrt{1-x^2}} = \frac{\pi}{3}\left(p\left(-\frac{1}{2}\sqrt{3}\right) + p\left(\frac{1}{2}\sqrt{3}\right)\right) \quad \forall p \in \mathcal{P}_5$

Sei $p(x) \in \mathcal{P}_5$, $p(x) = \sum_{k=0}^{5} a_k x^k$, $\int_{-1}^{1} \frac{x^k}{\sqrt{1-x^2}} = 0$ für ungeraden k

$\int_{-1}^{1} \frac{p(x)}{\sqrt{1-x^2}}\,dx = a_4 \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}} + a_2 \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\,dx + a_0 \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$

$\int_{-1}^{1} \frac{x^2\,dx}{\sqrt{1-x^2}} = \int_{0}^{\pi} \cos^4(t)\,dt$.

$x = \cos t$
$t = \arccos x$
$dx = -\sin t\,dt$

$$a_4 \int_{-1}^{1} \frac{x^4 dx}{\sqrt{1-x^2}} = a_4 \int_0^{\pi} \cos^4 t \, dt = \frac{3}{8} \pi a_4$$

$$a_2 \int_{-1}^{1} \frac{x^2 dx}{\sqrt{1-x^2}} = \ldots = \frac{1}{2} \pi a_2$$

$$a_0 \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \ldots = a_0 \pi$$

$$\Rightarrow \int_{-1}^{1} \frac{p(x) dx}{\sqrt{1-x^2}} = \frac{3}{8} \pi a_4 + \frac{1}{2} \pi a_2 + \pi a_0 = \frac{\pi}{3} \left(p\left(-\frac{1}{2}\sqrt{3}\right) + p(0) + p\left(\frac{1}{3}\sqrt{3}\right) \right) = \ldots$$

$$\ldots = \frac{3}{8} \pi a_4 + \frac{1}{2} \pi a_2 + \pi a_0$$