15. Blatt HM1

57 (a) $\displaystyle\int \frac{dx}{\sinh x \cosh x} = \left\{ \begin{array}{l} s = \sinh x \\ ds = dx \cdot \cosh x = dx \cdot \sqrt{1+s^2} \end{array} \right\} = \int \frac{ds}{\sqrt{1+s^2} \cdot s \cdot \sqrt{1+s^2}} = \int \frac{ds}{s(1+s^2)}$

PBZ $\displaystyle\frac{1}{s(1+s^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$, $1 = A\cdot(s^2+1) + (Bs+C)\cdot s$

$$\left. \begin{array}{l} s = 0 \rightsquigarrow 1 = A \\ s = 1 \rightsquigarrow 1 = 2A + B + C \\ s = -1 \rightsquigarrow 1 = 2A + B - C \end{array} \right\} \Rightarrow C = 0,\ B = -1$$

$$\Rightarrow \dots = \int \frac{1}{s} - \frac{s}{s^2+1}\, ds = \ln|s| - \frac{1}{2}\ln(s^2+1) + \gamma =$$

$$= \frac{1}{2} \ln \frac{s^2}{s^2+1} + \gamma = \frac{1}{2}\ln\left((\tanh x)^2\right) + \gamma = \ln|\tanh x| + \gamma \qquad \gamma \in \mathbb{R}$$

Probe: $\displaystyle\partial_x \dots = \frac{1}{\tanh x} \cdot \frac{1}{(\cosh x)^2} = \frac{1}{\sinh x \cdot \cosh x}$ ✓

Alternative (leichter): $\displaystyle\int \dots = \int \frac{e^{2x}\, dx \cdot 4}{(e^{2x}-1)(e^{2x}+1)} = \left\{ \begin{array}{l} y = e^{2x} \\ dy = 2e^{2x} dx \end{array} \right\} = \int \frac{\frac{1}{2}\, dy \cdot 4}{(y-1)(y+1)} =$

$$= \int \frac{1}{y-1} - \frac{1}{y+1}\, dy = \ln\left|\frac{y-1}{y+1}\right| + \gamma = \ln|\tanh x| + \gamma$$

(b) $\displaystyle\int \frac{x^7}{x^4+2}\, dx \underset{(1)}{=} \int \frac{x^7 + 2x^3 - 2x^3}{x^4+2}\, dx = \int x^3\, dx - \frac{1}{2}\int \frac{4x^3}{x^4+2}\, dx =$

Probe: $\displaystyle\partial_x = x^3 - \frac{2x^3}{x^4+2} = \frac{x^7+2x^3-2x^3}{x^4+2}$ ✓

$$= \frac{1}{4}x^4 - \frac{1}{2}\ln(x^4+2) + \gamma$$

Faktorisierung (PBZ) $x^4 + 2 = (x^2 + 2^{3/4}x + \sqrt{2})(x^2 - 2^{3/4}x + \sqrt{2})$ ist viel aufwendiger.

Bem.: $\displaystyle\boxed{\ln|g| = \int \frac{g'}{g}}$

(c) $\displaystyle\int \frac{1+\tan x}{\sin 2x}\, dx = \int \frac{1 + \tan\frac{y}{2}}{\sin y} \frac{dy}{2} = \left\{ \begin{array}{ll} t = \tan\frac{y}{2}, & \sin y = \frac{2t}{1+t^2} \\ dy = \frac{2\,dt}{1+t^2}, & \cos y = \frac{1-t^2}{1+t^2} \end{array} \right\} =$

$$= \int \frac{1+t}{2t} \frac{(1+t^2)}{2} \frac{2\,dt}{1+t^2} = \int \frac{1}{2t} + \frac{1}{2}\, dt = \frac{1}{2}\ln|t| + \frac{1}{2}t + \gamma =$$

$$= \frac{1}{2}\left(\ln|\tan x| + \tan x\right) + \gamma$$

Probe: $\displaystyle\partial_x \dots = \frac{1}{2}\left(\frac{1}{\tan x} \cdot \frac{1}{(\cos x)^2} + \frac{1}{(\cos x)^2}\right) = \frac{1}{2}\left(\frac{1}{sc} + \frac{1}{c \cdot c}\right) = \frac{1 + \frac{s}{c}}{2sc}$ ✓

(d) $f(x) = \dfrac{x^3+x^2}{x^4+x^3-3x^2-x+2} = \dfrac{x^2}{(x+2)(x-1)^2}$, denn: -1 raten (z.B.)

$x^4+x^3-3x^2-x+2 = (x+1)\cdot(x^3-3x+2)$

$$\begin{array}{l} x^4+x^3 \\ \hline \quad -3x^2 \\ \quad -3x^2-3x \\ \hline \qquad 2x+2 \\ \qquad // \end{array} \qquad \begin{array}{l} x^3-3x+2 = (x-1)\cdot\underbrace{(x^2+x-2)}_{=(x-1)(x+2)} \\ x^3-x^2 \\ \hline \quad x^2-3x \\ \quad x^2-x \\ \hline \qquad -2x+2 \\ \qquad // \end{array}$$

$(x+1)x^2$ = Zähler $\rightsquigarrow$ kürzen

$\Rightarrow f(x) = \dfrac{A}{x+2} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2} \quad \rightsquigarrow \quad A(x^2-2x+1) + B(x^2+x-2) + C(x+2) = x^2$

$\rightsquigarrow \quad x=1 \Rightarrow A\cdot 0 + B\cdot 0 + C\cdot 3 = 1 \quad \Rightarrow C = \frac{1}{3}$

$x=-2 \Rightarrow A\cdot 9 + B\cdot 0 + C\cdot 0 = 4 \quad \Rightarrow A = \frac{4}{9}$

$x=0 \Rightarrow A - 2B + 2C = 0 \quad \Rightarrow B = C + \frac{A}{2} = \frac{5}{9}$

$\Rightarrow \int f(x)\,dx = \frac{4}{9}\int \frac{1}{x+2}\,dx + \frac{5}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} =$

$= \frac{4}{9}\ln|x+2| + \frac{5}{9}\ln|x-1| - \frac{1/3}{(x-1)} + c$

Zu B: $\int_0^\infty f_k(x)\,dx = \lim_{M\to\infty} \left[a e^{-akx}\frac{1}{-ak} - b e^{-bkx}\frac{1}{-bk}\right]_0^M =$

$= \frac{1}{k}\lim_{M\to\infty}\left[e^{-kbM} - e^{-kaM} \underbrace{-e^{-kb\cdot 0} + e^{-ka\cdot 0}}_{-1+1}\right] = \frac{1}{k}\cdot(0-0) = 0$ $(\forall k\in\mathbb{N})$

$\Rightarrow B = \sum_{k=1}^\infty 0 = 0$

Zu A: $\sum_{k=1}^\infty f_k(x) \underset{\text{falls RS ex.}}{=} a\sum_1 (e^{-ax})^k - b\sum_1 (e^{-bx})^k \underset{x>0}{=} \dfrac{a e^{-ax}}{1-e^{-ax}} - \dfrac{b e^{-bx}}{1-e^{-bx}}$

$\sum_{k=1}^\infty f_k(0)$ ist nicht definiert.

$\Rightarrow \int^\infty \Big(\overbrace{\sum_{k=1}^\infty f_k(x)}^{=: f(x)\ (x>0)}\Big)dx = \int^1 \ldots dx + \int_1^\infty \ldots dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \ldots dx + \lim_{M\to\infty}\int_1^M \ldots dx$

15.B.2

$$\int_1^\infty \dots dx = \lim_{M\to\infty} \left[\ln|1-e^{-ax}| - \ln|1-e^{-bx}| \right]_1^M =$$

$$= \ln 1 - \ln 1 - \ln(1-e^{-a}) + \ln(1-e^{-b})$$

$$\int_0^1 \dots dx = \lim_{\varepsilon\to 0} \left[\ln(1-e^{-a}) - \ln(1-e^{-b}) - \ln(1-e^{-a\varepsilon}) + \ln(1-e^{-b\varepsilon}) \right]$$

$$\lim_{\varepsilon\to 0} \ln \frac{1-e^{-b\varepsilon}}{1-e^{-a\varepsilon}} \underset{\text{falls RS ex.}}{\overset{\text{l'H.}}{=}} \ln \lim_{\varepsilon\to 0} \frac{b e^{-b\varepsilon}}{a e^{-a\varepsilon}} = \ln \frac{b}{a} > 0$$

$$\Rightarrow A = \int_0^\infty f(x)\,dx = \left\{ \text{gilt auch „in 1 Rutsch"} \lim_{\substack{M\to\infty \\ \varepsilon\to 0}} \int_\varepsilon^M f(x)\,dx \right\} =$$

$$= \ln 1 - \ln 1 + \ln\frac{b}{a} = \ln\frac{b}{a} > 0 = B$$

9. Bekannt: $\int_1^\infty \frac{1}{x^\gamma}\,dx$ ex $(<\infty)$ g.d.w. $\gamma > 1$

$\int_0^1 \frac{1}{x^\gamma}\,dx$ ex gdw $\gamma < 1$

(a) (i) Sei $\beta > \alpha + 1$. Dann gilt für $x \geq 1$: $\frac{x^\alpha}{1+x^\beta} \leq \frac{x^\alpha}{x^\beta} = \frac{1}{x^{\beta-\alpha}}$

und damit konvergiert nach Majo $\int_1^\infty \frac{x^\alpha}{1+x^\beta}\,dx$

Sei $\beta \leq \alpha + 1$. Für $\alpha < -1$ gilt $\frac{x^\alpha}{1+x^\beta} \leq \frac{x^\alpha}{1} = \frac{1}{x^{-\alpha}}$ und das betrachtete Integral ex. nach Majo.

Für $\alpha = -1$: $\frac{x^\alpha}{1+x^\beta} = \frac{1}{x(1+x^\beta)} \geq \frac{1}{x} \cdot \frac{1}{1+1} = \frac{1}{2x} \Rightarrow$ nach Mino Divergenz

Für $\alpha > -1$: $\frac{x^\alpha}{1+x^\beta} \underset{x\geq 1}{\geq} \frac{x^\alpha}{1+x^{1+\alpha}}$ ~~$\underset{\substack{x\geq 1 \\ 1+\alpha>0}}{\geq} \frac{x^{-1}}{x^{1+\alpha}\,2} = \frac{1}{2}\frac{1}{x^{\alpha+2}} \Rightarrow$ Konv.~~

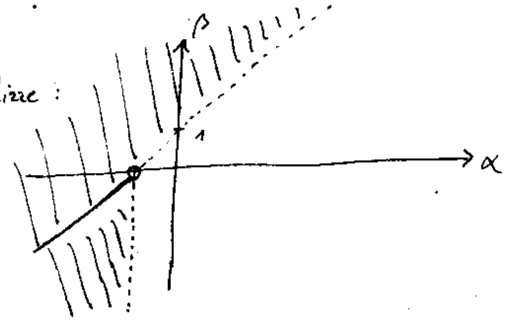
$$\Rightarrow \int_1^M \frac{x^\alpha}{1+x^\beta}\,dx \geq \int_1^M \frac{x^\alpha}{1+x^{1+\alpha}}\,dx = \frac{1}{1+\alpha} \ln\frac{1+M^{1+\alpha}}{2} \xrightarrow{M\to\infty} +\infty$$

$\Rightarrow$ Divergenz

15. B. 3

Fazit: $\int_1^\infty \frac{x^\alpha}{1+x^\beta}\,dx$ konv. genau dann, wenn $\beta > \alpha+1$ oder
($\beta \le \alpha+1$ und $\alpha < -1$)

Skizze:



(ii) $\frac{1-e^{-x}}{x} = \frac{e^{-x}-1}{-x} \xrightarrow{x\to 0} 1 \Rightarrow \exists\, a, b, \delta > 0$ $\forall x \in (0,\delta]: a \le \frac{1-e^{-x}}{x} \le b$

$\Rightarrow \frac{a}{x^{\alpha-1}} \le \frac{1-e^{-x}}{x^\alpha} \le \frac{b}{x^{\alpha-1}}$ für $x \in (0,\delta]$ mit gewissen pos. a, b, δ

Mino $\Rightarrow \int_0^{\delta 1} \frac{1-e^{-x}}{x^\alpha}\,dx$ div. für $\alpha \ge 2$, Majo $\Rightarrow \int_0^{\delta 1}$ konv. für $\alpha < 2$

Für $x \to \infty$ gilt $1-e^{-x} \to 1 \Rightarrow \exists$ positive $\tilde a, \tilde b, M$ mit $\tilde a \le 1-e^{-x} \le \tilde b$ für alle $x \ge M$ $\Leftrightarrow \exists\, \tilde a, \tilde b, M > 0$, $\forall x \ge M: \frac{\tilde a}{x^\alpha} \le \frac{1-e^{-x}}{x^\alpha} \le \frac{\tilde b}{x^\alpha}$

$\Rightarrow \left(\int_1^\infty \frac{1-e^{-x}}{x^\alpha}\,dx \text{ konv.} \Leftrightarrow \alpha > 1 \right)$

$\Rightarrow \int_0^\infty \frac{1-e^{-x}}{x^\alpha}\,dx$ konv. genau für $1 < \alpha < 2$

(b) (i) $\alpha > 0$, $\beta \neq 0$: $\int_0^M e^{-\alpha x}\cos\beta x\,dx = \left[\frac{1}{\beta}\sin(\beta x)e^{-\alpha x}\right]_0^M + \alpha\int_0^M \frac{1}{\beta}\sin\beta x\, e^{-\alpha x}\,dx =$

$$= \frac{\sin\beta M}{\beta}e^{-\alpha M} + \frac{\alpha}{\beta}\left[-\frac{1}{\beta}\cos(\beta x)e^{-\alpha x}\right]_0^M - \frac{\alpha}{\beta}\int_0^M \alpha\frac{\cos\beta x}{\beta}e^{-\alpha x}\,dx$$

$$= \left(\frac{\sin\beta M}{\beta}e^{-\alpha M} - \frac{\alpha}{\beta^2}\cos(\beta M)e^{-\alpha M} + \frac{\alpha}{\beta^2}\right)\frac{1}{1+\frac{\alpha^2}{\beta^2}} \xrightarrow{M\to\infty} \frac{\alpha}{\alpha^2+\beta^2}$$

$\alpha > 0 = \beta$: $\int_0^M e^{-\alpha x}\,dx = \frac{1}{\alpha} - \frac{e^{-\alpha M}}{\alpha} \xrightarrow{M\to\infty} \frac{1}{\alpha} = \frac{\alpha}{\alpha^2+\beta^2}$

$\alpha = \beta = 0$: $\int_0^M 1\,dx = M \xrightarrow{M\to\infty} \infty$

$\alpha = 0 \neq \beta$: $\int_0^M \cos\beta x\,dx = \frac{\sin\beta M}{\beta} - 0 \xrightarrow{M\to\infty}$ div.

$\alpha < 0 = \beta$: $\int_0^M e^{-\alpha x} dx = \frac{1}{\alpha} - \frac{1}{\alpha} e^{|\alpha| M} \xrightarrow{M \to \infty} +\infty \quad (\alpha < 0 !)$

$\alpha < 0 \neq \beta$: $\int_0^M e^{-\alpha x} \cos \beta x \, dx = \ldots s.o. \ldots = \frac{1}{\alpha^2 + \beta^2} \left(\alpha + e^{|\alpha| M} \left(\beta \sin \beta M - \alpha \cos \beta M \right) \right)$

divergiert für $M \to \infty$: $M_k := \frac{2\pi}{\beta} \cdot k \; (k \in \mathbb{N}) \Rightarrow e^{|\alpha| M_k} \cdot (0 - \alpha \cdot 1) \xrightarrow{k \to \infty} +\infty$

$\tilde{M}_k := M_k + \frac{\pi}{\beta} \Rightarrow e^{|\alpha| M_k} (0 + \alpha \cdot 1) \xrightarrow{k \to \infty} -\infty$

Fazit: $\int_0^\infty e^{-\alpha x} \cos \beta x \, dx$ ex. genau für $\alpha > 0$ und hat den Wert $\frac{\alpha}{\alpha^2 + \beta^2}$

[Für $\beta = 0 \geq \alpha$ liegt bestimmte Divergenz gegen $+\infty$ vor.]

(ii) $\int_e^M \frac{dx}{x (\ln x)^\alpha} \underset{\alpha \neq 1}{=} \left[\frac{1}{-\alpha + 1} (\ln x)^{-\alpha + 1} \right]_e^M = \left((\ln M)^{1-\alpha} - 1 \right) \frac{1}{1-\alpha}$

$\Rightarrow$ für $\alpha < 1$ divergiert das Integral bestimmt gegen $+\infty$, für $\alpha > 1$ konvergiert es gegen $\frac{1}{\alpha - 1} > 0$.

Für $\alpha = 1$: $\int_e^M \frac{dx}{x \ln x} \leq \int_e^M \frac{dx}{x \cdot 1} \xrightarrow{M \to \infty} \infty$

Probe: $\partial_x = \frac{1}{2} \frac{\frac{1}{c^2} \cdot c + t \cdot s}{(\cos x)^2} + \frac{1}{2} \frac{\cancel{1 + \tan \frac{x}{2}}}{1 + \tan \frac{x}{2}} \cdot \frac{1}{2} \left(1 + \tan^2 \frac{x}{2} \right) + \frac{1}{2} \frac{1}{1 - \tan \frac{x}{2}} \cdot \frac{1}{2} \cdot \left(1 + \tan^2 \frac{x}{2} \right)$

$= \frac{1 + (\sin x)^2}{(\cos x)^3 \cdot 2} + \underbrace{\frac{1 + \tan^2 \frac{x}{2}}{4} \frac{1 - \tan \frac{x}{2} + 1 + \tan \frac{x}{2}}{(1 + \tan \frac{x}{2})(1 - \tan \frac{x}{2})}}_{\frac{1}{2 \cos^2 \frac{x}{2}} \cdot \frac{1}{1 - t^2} = \frac{1}{2} \frac{1}{c^2 - s^2} = \frac{1}{2 \cos (2 \cdot \frac{x}{2})}} = \frac{1}{(\cos x)^3} - \frac{1}{2 \cos x} + \frac{1}{2 \cos x}$

(d) $\int \frac{dx}{\sin^{3/4} x \cos^{5/4} x} = \left\{ \begin{array}{ll} t = \tan x, & \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2}, & \cos x = \frac{1}{\sqrt{1+t^2}} \end{array} \right\} = \int \frac{1}{\left(\frac{t^3}{\sqrt{1+t^2}^3} \cdot \frac{1}{\sqrt{1+t^2}^5} \right)^{1/4}} \cdot \frac{dt}{1+t^2}$

$= \int \frac{dt}{t^{3/4}} = 4 t^{1/4} + \gamma = 4 (\tan x)^{1/4} + \gamma$

Probe: $\partial_x \ldots = (\tan x)^{-\frac{3}{4}} \frac{1}{\cos^2 x} = \frac{1}{s^{\frac{3}{4}} \cdot c^{\frac{5}{4}}}$ ✓

60 (a) $\int (x^5+x)\, e^{-x^2}\, dx = \left\{ \begin{array}{l} x^2 = t \\ 2x\,dx = dt \end{array} \right\} = \frac{1}{2} \int (t^2+1) e^{-t}\, dt$ $\quad$ 15.3.5

Ansatz: $(At^2+Bt+C)\, e^{-t} \xrightarrow{\partial_t} e^{-t}(2At + B - At^2 - Bt - C) \overset{!}{=} e^{-t}(t^2+1)$

$$\rightsquigarrow \left(\begin{array}{ccc|c} -1 & 0 & 0 & 1 \\ 2 & -1 & & 0 \\ & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 \\ & 0 & -1 & 3 \end{array}\right) \rightsquigarrow \begin{array}{l} A = -1 \\ B = -2 \\ C = -3 \end{array}$$

$$\Rightarrow \dots = \frac{1}{2}(-t^2 - 2t - 3)\, e^{-t} + \gamma = -\frac{1}{2}(x^4 + 2x^2 + 3)\, e^{-x^2} + \gamma$$

Probe: $\partial_x \dots = e^{-x^2}\left(-2x^3 - 2x + x(x^4 + 2x^2 + 3)\right) = e^{-x^2}(x^5 + x)$ ✓

(b) $f(x) = \frac{x^4 - x^3 - 3x - 1}{x^4 + 4x^2 + 3} \underset{(1)}{=} \frac{x^4 + 4x^2 + 3 - x^3 - 4x^2 - 3x - 4}{x^4 + 4x^2 + 3} \underset{(2)}{=} 1 - \frac{x^3 + 4x^2 + 3x + 4}{(x^2+1)(x^2+3)}$

$\overset{(4)}{=} 1 - \frac{Ax+B}{x^2+1} - \frac{Cx+D}{x^2+3}$; (5) $x^3 + 4x^2 + 3x + 4 = (Ax+B)(x^2+3) + (Cx+D) \cdot (x^2+1)$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 4 \\ 3 & 0 & 1 & 0 & 3 \\ & 3 & & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ & 1 & & 1 & 4 \\ & 0 & -2 & 0 & 0 \\ & 0 & 0 & -2 & -8 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ & 1 & & & 0 \\ & & 1 & & 0 \\ & & & 1 & 4 \end{array}\right) \quad \begin{array}{l} 1 = A \\ 0 = B \\ 0 = C \\ 4 = D \end{array}$$

(6) $\int f(x)\, dx = \int \left(1 - \frac{x+0}{x^2+1} - \frac{0 \cdot x + 4}{x^2+3}\right) dx = \gamma + x - \frac{1}{2}\ln(x^2+1) -$

$- \arctan \frac{x}{\sqrt{3}} \cdot \frac{4}{3} \cdot \sqrt{3}$

Probe: $\partial_x \dots = 1 + \frac{-x}{x^2+1} - \frac{4}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{1 + \left(\frac{x}{\sqrt{3}}\right)^2} = 1 - \frac{x}{x^2+1} - \frac{4}{x^2+3} =$

$= \frac{x^4 + 4x^2 + 3 - x^3 - 3x - 4x^2 - 4}{(x^2+1)(x^2+3)}$ ✓

(c) $\int \frac{dx}{(\cos x)^3} = \int \underbrace{\frac{1}{\cos x}}_{v} \cdot \underbrace{\frac{1}{(\cos x)^2}}_{u'} dx = \underbrace{\frac{1}{\cos x}}_{v} \cdot \underbrace{\tan x}_{u} - \int \tan x \frac{\sin x}{(\cos x)^2}\, dx =$

$= \frac{\tan x}{\cos x} - \int \frac{1 - (\cos x)^2}{(\cos x)^3}\, dx = \frac{\tan x}{\cos x} + \int \frac{dx}{\cos x} - \int \frac{dx}{(\cos x)^3} =$

$= \left(\frac{\tan x}{\cos x} + \int \frac{dx}{\cos x}\right) \cdot \frac{1}{2} \underset{\text{Vol.}}{=} \frac{1}{2}\left(\frac{\tan x}{\cos x} + \ln\left|\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right|\right) + \gamma$