(a)

$$\int \tanh x \cdot \ln(\cosh x)\,dx = \left\{ u=\cosh x,\ \frac{du}{dx}=\sinh x \right\} = \int \frac{\ln u}{u}\,du = \frac{1}{2}(\ln u)^2 + c =$$

$$= \frac{1}{2}\left(\ln(\cosh x)\right)^2 + c \quad \text{wegen } (f^2)' = 2ff',\ \frac{1}{u} = \frac{d}{du}\ln u \ \left(\text{oder } v=\ln u,\ \frac{dv}{du}=\frac{1}{u}\right)$$

Probe durch differenzieren

(b)

$$\int \frac{x}{(\cos x)^2}\,dx = \left\{ u'=\frac{1}{(\cos x)^2}=(\tan x)',\ v=x \right\} = x\cdot\tan x - \int 1\cdot\tan x\,dx = x\cdot\tan x + \ln(\cos x) + c$$

$\cancel{x \notin \frac{\pi}{2}+\pi\mathbb{Z}}$ Probe: $\left(x\tan x + \ln(\cos x)\right)' = \tan x + \frac{x}{(\cos x)^2} + \frac{1}{\cos x}\cdot(-\sin x) = \frac{x}{(\cos x)^2}$

$|x| < \frac{\pi}{2}$

(c)

$$\int x^3 \sin x\,dx = \left\{ u'=\sin x,\ v=x^3 \right\} = -x^3\cos x + \int \cos x \cdot 3x^2\,dx = \left\{ u'=\cos x,\ v=x^2 \right\} =$$

$$= -x^3\cos x + 3x^2\sin x - \int 6x\sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\int 1\cdot\cos x\,dx$$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + c$$

Probe: $(\dots)' = x^3\sin x - 3x^2\cos x + 3x^2\cos x + 6x\sin x + 6\cos x - 6x\sin x - 6\cos x$ ✓

(d)

$$\int \frac{(\ln x)^4 - 1}{x\left((\ln x)^3+1\right)}\,dx = \left\{ y=\ln x,\ dy=\frac{1}{x}dx \right\} = \int \frac{y^4-1}{y^3+1}\,dy = \int y - \frac{y+1}{y^3+1}\,dy =$$

$$= \frac{1}{2}y^2 - \int \frac{y+1}{(y+1)(y^2-y+1)}\,dy = \frac{1}{2}y^2 - \int \frac{dy}{\left(y-\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{2}y^2 - \frac{4}{3}\int \frac{dy}{1+\left(\frac{(y-\frac{1}{2})\cdot 2}{\sqrt{3}}\right)^2} =$$

$$= \left\{ z=\frac{2y-1}{\sqrt{3}},\ dz=\frac{2}{\sqrt{3}}dy \right\} = \frac{1}{2}y^2 - \frac{4}{3}\int \frac{dz}{1+z^2}\cdot\frac{\sqrt{3}}{2} = \frac{1}{2}y^2 - \frac{2}{\sqrt{3}}\arctan z +$$

$$= \frac{1}{2}(\ln x)^2 - \frac{2}{\sqrt{3}}\arctan\frac{2\ln x - 1}{\sqrt{3}} + c$$

Probe: $(\dots)' = \frac{1}{x}\ln x - \frac{2}{\sqrt{3}}\cdot\frac{1}{1+\left(\frac{2\ln x-1}{\sqrt{3}}\right)^2}\cdot\frac{2}{\sqrt{3}x} = \frac{\ln x}{x} -$

$$-\frac{4}{3x}\frac{1\cdot 3}{3+4(\ln x)^2-4\ln x+1} = \frac{\ln x}{x} - \frac{1}{x\left((\ln x)^2-\ln x+1\right)}\cdot\frac{\ln x+1}{\ln x+1} = \frac{\ln x\left((\ln x)^3+1\right)-\ln x-}{x\left((\ln x)^3+1\right)}$$

Bem.: Nimmt man als Definitionsbereich $(0,\frac{1}{e}) \cup (\frac{1}{e},\infty)$ statt nur eines der Intervalle, dann muß man für die 2 Intervalle auch 2 Konstanten wählen: $\int\dots = \dots + \begin{cases} c_1 & \text{für } (0,\frac{1}{e}) \\ c_2 & \text{für } x > \frac{1}{e} \end{cases}$

(a) $$\sum_{k=1}^{n} \frac{k}{n^2+k^2} = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{k/n}{1+\left(\frac{k}{n}\right)^2} \xrightarrow{n\to\infty} \int_0^1 \frac{x}{1+x^2}\,dx = \int_0^1 \frac{1}{2}\left(\ln(1+x^2)\right)' dx =$$

$$= \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{1}{2}\ln 2$$

(b) $$\frac{1}{n}\prod_{k=1}^{n}(n+k)^{1/n} = \prod_{k=1}^{n}\frac{(n+k)^{1/n}}{n^{1/n}} = \prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{1/n} = \exp\left(\ln\prod_{k=1}^{n}\cdots\right) = \exp\left(\sum_{k=1}^{n}\frac{1}{n}\ln\left(1+\frac{k}{n}\right)\right)$$

$$\xrightarrow{n\to\infty} \underset{\text{stetig}}{\exp}\left(\int_0^1 \ln(1+x)\,dx\right) = \exp\left(\left[(1+x)\ln(1+x)-(1+x)\right]_0^1\right) = e^{2\ln 2 - 2 - 1\ln 1 + 1}$$

$$= \frac{4}{e}$$

(c) $$\sum_{k=1}^{n}\frac{1}{n\alpha+k\beta} = \sum_{k=1}^{n}\frac{1}{n}\,\frac{1}{\alpha+\frac{k}{n}\beta} \xrightarrow{n\to\infty} \int_0^1 \frac{1}{\alpha+\beta x}\,dx = \int_0^1\left(\frac{d}{dx}\ln(\alpha+\beta x)\right)\frac{1}{\beta}\,dx =$$

$$= \frac{1}{\beta}\left(\ln(\alpha+\beta)-\ln\alpha\right) = \frac{1}{\beta}\ln\left(1+\frac{\beta}{\alpha}\right)$$

Falls $\beta = 0$: $\sum_{k=1}^{n}\frac{1}{n\alpha} = \frac{1}{\alpha} \xrightarrow{n\to\infty} \frac{1}{\alpha}$

Falls $\alpha = 0$: $\sum_{k=1}^{n}\frac{1}{k\beta}$ ist divergent $(\to\infty)$

war nicht verlangt

Falls beide <0: $-\sum_{k=1}^{n}\frac{1}{n|\alpha|+k|\beta|} \xrightarrow{n\to\infty} -\frac{1}{|\beta|}\ln\left(1+\left|\frac{\beta}{\alpha}\right|\right) = \frac{1}{\beta}\ln\left(1+\frac{\beta}{\alpha}\right)$

Falls genau eines <0: ~~oBdA~~ $\operatorname{sgn}\alpha \cdot \sum_{k=1}^{n}\frac{1}{n|\alpha|-k|\beta|}$ ist nicht definiert, falls $\frac{k}{n} = \left|\frac{\alpha}{\beta}\right| \in \mathbb{Q}$.

Falls $\left|\frac{\alpha}{\beta}\right| \in \mathbb{R}\setminus\mathbb{Q}$, ergibt sich wie oben $-\frac{1}{|\beta|}\ln\left(1-\left|\frac{\beta}{\alpha}\right|\right)\cdot\operatorname{sgn}\alpha$, aber nur für $|\alpha|>|\beta|$

~~Für $|\alpha|<|\beta|$ divergiert die Summe für $n\to\infty$, da $\frac{k}{n}$ für „günstiges" k beliebig nahe an $\left|\frac{\alpha}{\beta}\right|$ kommt und der entsprechende Summand $\frac{1}{n|\alpha|-k|\beta|}$ dann beliebig groß.~~

Hm, das ist kein Argument, da ja auch viele negative Summanden existieren, die das hinbiegen könnten. Ich glaube trotzdem an Divergenz, weil „des isch halt so" — fällt euch etwas ein?

ⓓ $$\lim_{n\to\infty}\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{k=1}^{2n}\frac{k^j}{n^{j+1}} \overset{(*)}{=} \lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{n}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{k}{n}\right)^j = \lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{n}e^{k/n} \underset{(+)}{=}$$

$$= \int_0^2 e^x\,dx = e^2 - e^0 = e^2 - 1$$

Die „Umordnung" bei (*) ist erlaubt, weil das eine eine endliche Summe ist ($\sum_{k=1}^{2n}$) und auf der RS alles konvergiert (für jedes k konv. die Reihen $\sum_{j=0}^{\infty}\dots$).

(+) gilt, weil die Summe links eine Riemann-Summe (oder Ober-Summe) für e^x in $[0,2]$ ist und $(x \mapsto e^x) \in R[0,2]$ bekannt ist. (So lautet auch die Argumentation bei den anderen Aufgaben.)

$\boxed{55}$

ⓒ $$\int_0^1 \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \left\{ \cancel{y=\sqrt[6]{x},\ \frac{dy}{dx}=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3}\frac{1}{y^2}} \atop y = x^{\frac{1}{6}},\ \frac{dy}{dx}=\frac{1}{6}x^{-\frac{5}{6}}=\frac{1}{6y^5} \right\} = \int_0^{\sqrt[6]{1}} \frac{y^3}{1+y^2}6y^5\,dy =$$

$$=6\int_0^1 \frac{y^8+y^6-y^6+-y^4+y^4+y^2-y^2-1+1}{1+y^2}\,dy = 6\int_0^1\left(y^6-y^4+y^2-1+\frac{1}{1+y^2}\right)dy =$$

$$= 6\left[\frac{1}{7}y^7-\frac{1}{5}y^5+\frac{1}{3}y^3-y+\arctan y\right]_0^1 = 6\cdot\left(\frac{1}{7}-\frac{1}{5}+\frac{1}{3}-1+\frac{\pi}{4}\right) = \frac{3}{2}\pi - \frac{12}{35} - 4 = \frac{3}{2}\pi - \frac{152}{35}$$

ⓑ $$\int_2^4 \frac{1}{x\sqrt{x^2-1}}\,dx = \left\{ x=\cosh y \atop \sqrt{x^2-1}=\sinh y = \frac{dx}{dy} \right\} = \int_{\operatorname{arcosh} 2}^{\operatorname{arcosh} 4} \frac{\sinh y\,dy}{\cosh y\,\sinh y} = \int_a^b \frac{2\,dy}{e^y+e^{-y}} = \left\{ t=e^y \atop \frac{dt}{dy}=e^y=t \right\} =$$

$$= \int_{e^a}^{e^b} \frac{2}{t+\frac{1}{t}}\,\frac{dt}{t} = 2\int_{e^a}^{e^b}\frac{dt}{1+t^2} = 2\left(\arctan e^b - \arctan e^a\right)$$

$$\operatorname{arcosh} x = \ln\left(x+\sqrt{x^2-1}\right) \Rightarrow e^a = 2+\sqrt{2^2-1} = 2+\sqrt{3},\ e^b = 4+\sqrt{15}$$

$$\Rightarrow \text{Ergebnis} = 2\left(\arctan\left(4+\sqrt{15}\right) - \arctan\left(2+\sqrt{3}\right)\right)$$

$$\int_{\frac{1}{2}\sqrt{2}}^1 x\arcsin x\,dx = \left\{ u'=x,\ v=\arcsin x \atop u=\frac{1}{2}x^2,\ v'=\frac{1}{\sqrt{1-x^2}} \right\} = \left[\frac{1}{2}x^2\arcsin x\right]_{\frac{1}{\sqrt{2}}}^1 - \frac{1}{2}\int_{\frac{1}{\sqrt{2}}}^{1}\frac{x^2}{\sqrt{1-x^2}}\,dx = \left\{ x=\cos t \atop \frac{dx}{dt}=-\sin t \right\}$$

$$= \left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2\sqrt{2}^2}\cdot\frac{\pi}{4}\right) + \frac{1}{2}\int_{\frac{\pi}{4}}^{0}\frac{(\cos t)^2\sin t\,dt}{\sin t} = \frac{3}{16}\pi + \frac{1}{2}\left[\sin t\cos t\Big|_{\frac{\pi}{4}}^{0} + \int_{\frac{\pi}{4}}^{0}(\sin t)^2\,dt\right] =$$

$\sin t > 0$ in $\cancel{[\tfrac{1}{2}\sqrt{2},1]}\ [0, \frac{\pi}{4}]$

14.B.4

(c) $\displaystyle\int_0^{1/\sqrt{2}} x \arcsin x \, dx = \left\{\begin{matrix} u' = x, & v = \arcsin x \\ u = \frac{1}{2}x^2, & v' = \frac{1}{\sqrt{1-x^2}} \end{matrix}\right\} = \left[\frac{1}{2}x^2 \arcsin x\right]_0^{1/\sqrt{2}} - \frac{1}{2}\int_0^{1/\sqrt{2}} x^2 \frac{dx}{\sqrt{1-x^2}}$

$$\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}}\, dx = \left\{\begin{matrix} x = \sin t \\ \frac{dx}{dt} = \cos t \end{matrix}\right\} = \int_0^{\pi/4} \frac{(\sin t)^2 \cos t}{\cos t}\, dt = -\cos t \sin t + \int (\cos t)^2\, dt =$$

$$= -\cos t \sin t + \int 1-(\sin t)^2\, dt = \frac{1}{2}\left(-\cos t \sin t + t\right)\Big|_0^{\pi/4}$$

$$\Rightarrow \cancel{\left[\tfrac{1}{2}x^2 \arcsin x\right]_0^{1/\sqrt{2}} - \tfrac{1}{2}\cdot\tfrac{1}{2}\left[-x\right.}$$

$$\int_0^{1/\sqrt{2}} x \arcsin x \, dx = \frac{1}{4}\arcsin\left(\frac{1}{2}\sqrt{2}\right) - 0\cdot\arcsin 0 - \frac{1}{2}\cdot\frac{1}{2}\left(-\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{16} + \frac{1}{8} - \frac{\pi}{16} = \frac{1}{8}$$

(d) $\displaystyle\int_{-\pi/4}^{\pi/4} \tanh(e \tanh x)\, dx = \left\{\begin{matrix} y = -x \\ dy = -dx \end{matrix}\right\} = -\int_{\pi/4}^{-\pi/4} \tanh(e\tanh(-y))\, dy \underset{\text{tanh ist ungerade}}{=} + \int_{\pi/4}^{-\pi/4} \tanh(e \tanh y)\, dy$

$$= -\int_{-\pi/4}^{\pi/4} \tanh(e \tanh y)\, dy = 0 \quad (a = -a \Leftrightarrow a = 0)$$

$\boxed{56}$ $f \in C[-1,1] \subseteq R[-1,1]$, $\exists M > 0: \forall x \in [-1,1]: |f(x)| \le M$

$g_n: x \mapsto \frac{f(x)}{1+n^2x^2}$ ist stetig für jedes $n \in \mathbb{N}$ und daher Rib ($\forall n: g_n \in R[-1,1]$).

1. Fall f konstant $\Rightarrow \forall x: f(x) = f(0)$

$$\Rightarrow \lim_{n\to\infty} \int_{-1}^{1} n\, g_n(x)\, dx = f(0)\cdot \lim \int_{-1}^{1} \frac{n\, dx}{1+n^2x^2} = f(0) \lim_{n\to\infty} \left[\arctan(nx)\right]_{-1}^{1} =$$

$$= f(0)\cdot \lim\left(\arctan n - \arctan(-n)\right) = f(0)\cdot 2 \lim_{n\to\infty} \arctan n = f(0)\cdot 2 \cdot \frac{\pi}{2}$$

2. Fall $f(0) = 0$

f stetig $\Rightarrow \forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0: \forall |x| \le \delta: |f(x)| < \varepsilon$

Sei $\delta < |x| \le 1 \Rightarrow |n\, g_n(x)| \le \frac{M}{\frac{1}{n} + n x^2} \le \frac{M}{n x^2} \le \frac{M}{\delta^2}\cdot\frac{1}{n} \xrightarrow{n\to\infty} 0$

d.h. $n\, g_n \xrightarrow{n\to\infty} 0$ glm. auf $[-1,1] \setminus (-\delta, \delta)$

$$\Rightarrow \left| \int_{-1}^{1} n g_n(x)\, dx \right| \leq \underbrace{\left| \int_{-1}^{-\delta} n g_n(x)\, dx \right| + \left| \int_{\delta}^{1} n g_n(x)\, dx \right|}_{\xrightarrow{n\to\infty} 0 \text{, da Integrand glm. } \to 0} + \left| \int_{-\delta}^{\delta} n g_n(x)\, dx \right|$$

$$\left| \int_{-\delta}^{\delta} n g_n(x)\, dx \right| \leq \int |n g_n| \leq \int_{-\delta}^{\delta} \frac{n\varepsilon}{1+n^2x^2}\, dx \leq \int_{-1}^{1} \frac{n\varepsilon}{1+n^2x^2}\, dx \xrightarrow[\text{Fall 1}]{n\to\infty} \varepsilon\pi$$

$$\Rightarrow \forall \varepsilon > 0 \ \lim \left| \int_{-1}^{1} n g_n(x)\, dx \right| \leq \varepsilon\pi \quad \Rightarrow \quad \lim \int_{-1}^{1} n g_n(x)\, dx = 0$$

Fall 3 f bel.

$$\lim_{n\to\infty} n \int_{-1}^{1} \frac{f(x)}{1+n^2x^2}\, dx = \lim_{n\to\infty} \left(n \int_{-1}^{1} \frac{f(x)-f(0)}{1+n^2x^2}\, dx + n \int_{-1}^{1} \frac{f(0)}{1+n^2x^2}\, dx \right) =$$

$$= 0 + \pi \cdot f(0)$$ nach den Fällen 2 bzw. 1.

∎