$\boxed{25.}$ K (1) ⓐ $\sum_{k=1}^{\infty}\left(\frac{1}{2}+\frac{1}{2k}\right)^k x^{k-1} = \cancel{\sum_{j=0}^{\infty}\left(1+\frac{1}{j+1}\right)^{j+1}\left(\frac{x}{2}\right)^j\cdot\frac{1}{2}}$ $\underline{\text{WK } \sqrt[k]{\left(\frac{1}{2}+\frac{1}{2k}\right)^k} = \frac{1}{2}+\frac{1}{2k} \to \frac{1}{2} = \frac{1}{R}}$

$\cancel{\text{QK}: \left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+3}{n+2}\right)^{n+2}\frac{1}{2^{n+2}}\left(\frac{n+1}{n+2}\right)^{n+1}\cdot 2^{n+1} = \frac{1}{2}\left(1+\frac{1}{n+2}\right)^{n+2}\cdot\left(1+\frac{1}{n+1}\right)^{-(n+1)} \xrightarrow{n\to\infty} \frac{1}{2}e\cdot\frac{1}{e} =}$

$\Rightarrow R = 2$ (Konvergenzradius)

$x=\pm 2$: $\left(1+\frac{1}{j+1}\right)^{j+1}\cancel{1}\cdot\frac{1}{2} \xrightarrow{j\to\infty} \frac{1}{2}e \neq 0 \Rightarrow$ Reihe beidemale divergent. Also Konv. $\Leftrightarrow |x|<$

ⓑ $\sum_{k=0}^{\infty}\frac{x^{4k+1}}{(4+(-1)^k)^{3k}} = x\cdot\sum_{k=0}^{\infty}\frac{(x^4)^k}{(4+(-1)^k)^{3k}}$; setze $y := x^4$ u. betrachte $\sum_{k=0}^{\infty}\frac{y^k}{(4+(-1)^k)^{3k}}$

WK: $\sqrt[n]{|a_n|} = \frac{1}{(4+(-1)^n)^3} = \begin{cases}\frac{1}{125}, & n \text{ gerade}\\ \frac{1}{27}, & n \text{ unger.}\end{cases} \Rightarrow R_y = 27 \Rightarrow R = R_x = \sqrt[4]{27}$

$x = \sqrt[4]{27}$: $\sum_{k=0}^{\infty}\frac{27^k}{(4+(-1)^k)^{3k}} = \sum_{n=0}^{\infty}\frac{27^{2n}}{125^{2n}} + \sum_{n=0}^{\infty}\frac{27^{2n+1}}{27^{2n+1}} \Rightarrow$ Divergenz

ebenso $x = -\sqrt[4]{27}$. Also Konvergenz $\Leftrightarrow |x| < \sqrt[4]{27}$

ⓒ $\sum_{k=0}^{\infty} a^{k^2}x^k$. $\sqrt[k]{|a^{k^2}|} = |a|^{k^2/k} = |a|^k \xrightarrow{k\to\infty} \begin{cases}\infty & \text{für } |a|>1\\ 1 & |a|=1\\ 0 & |a|<1\end{cases}$

$\Rightarrow R = \begin{cases}0 & \text{für } |a|>1\\ 1 & |a|=1\\ \infty & |a|<1\end{cases}$

Sei $|a| = 1 = |x| \Rightarrow \sum_{k=0}^{\infty} a^{k^2}x^k = \sum_{k=0}^{\infty}(\pm 1)^k$, + für gleiches VZ von a und x, − für verschiedenes VZ. Beidemale Divergenz, da $(\pm 1)^k \not\to 0$.

$\Rightarrow$ (Konvergenz $\Leftrightarrow$ $x\in\mathbb{R}$ falls $|a|<1$, $|x|<1$ falls $|a|=1$, $x=0$ falls $|a|>1$)

ⓓ $\sum_{k=0}^{\infty}\frac{x^{k!}}{\sqrt{k+1}} = \sum_{n=0}^{\infty} a_n x^n$ mit $a_n = \begin{cases}\frac{1}{\sqrt{k+1}} & \text{für } n = k!\\ 0 & \text{sonst}\end{cases}$

$\limsup_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{k\to\infty}\sqrt[k!]{a_{k!}} = 1$, denn: $\sqrt[k!]{\frac{1}{\sqrt{k+1}}} \leq \sqrt[k!]{1} = 1$ und

$\sqrt[k!]{\frac{1}{\sqrt{k+1}}} \geq \sqrt[k]{\frac{1}{\sqrt{k+1}}} \geq \sqrt[k]{\frac{1}{\sqrt{2k}}} = \left(\frac{1}{\sqrt[k]{k}}\,\frac{1}{\sqrt[k]{2}}\right)^{1/2} \xrightarrow{k\to\infty} \left(\frac{1}{1}\cdot\frac{1}{1}\right)^{1/2} = 1$

$\Rightarrow R = 1$

Randpunkte $x=1$: Divergenz, da $\frac{1}{\sqrt{k+1}} \geq \frac{1}{k+1}$.

$x=-1$: $(-1)^{k!} = +1$ für $k\geq 2$, also Divergenz wie bei $x=1$

Ergo: Konvergenz $\Leftrightarrow |x|<1$

(2) ⓐ QK: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)!}{n!} \dfrac{3\cdots(2n+1)}{3\cdots(2n+1)(2n+3)} = \dfrac{n+1}{2n+3} \xrightarrow{n\to\infty} \dfrac{1}{2} = \dfrac{1}{R}$

oder WK: $\sqrt[n]{a_n} = \sqrt[n]{\dfrac{1\cdot 2\cdots n}{3\cdot 5\cdots(2n+1)}} < \sqrt[n]{\dfrac{1}{2}\cdot\dfrac{1}{2}\cdots\dfrac{1}{2}} = \dfrac{1}{2}$

$> \sqrt[n]{1\cdot\underbrace{\dfrac{1}{2}\cdots\dfrac{1}{2}}_{\left(\frac{1}{2}\right)^{n-1}}\cdot\dfrac{1}{2n+1}} = \dfrac{1}{2}\sqrt[n]{\dfrac{1}{n+\frac{1}{2}}} > \dfrac{1}{2}\sqrt[n]{\dfrac{1}{2n}} \to \dfrac{1}{2}$

ⓑ QK: $\dfrac{a_{n+1}}{a_n} = \dfrac{(2n+1)^{2n+1}\, 2^{2n}(2n)!}{2^{2n+2}(2n+2)!\,(2n-1)^{2n-1}} = \left(\dfrac{2n+1}{2n-1}\right)^{2n}(2n-1)\,\dfrac{1}{4(2n+2)} =$

$= \left(\dfrac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\right)^{2n} \dfrac{2-\frac{1}{n}}{1+\frac{1}{n}}\cdot\dfrac{1}{8} \xrightarrow{n\to\infty} \dfrac{2}{8}\cdot\dfrac{e}{1/e} = \dfrac{e^2}{4} \Rightarrow R = \dfrac{4}{e^2}$

[WK: $\sqrt[n]{a_n} = \dfrac{1}{4}\left(\dfrac{(2n-1)}{(2n)!^{1/2n}}\,\dfrac{1}{(2n-1)^{1/2n}}\right)^2 = \dfrac{1}{4}\left(\underbrace{\dfrac{2n}{(2n)!^{1/2n}}}_{\to e} - \underbrace{\dfrac{1}{\sqrt[2n]{(2n)!}}}_{\to 0}\right)^2 \cdot$

$\left(\underbrace{\left(1-\dfrac{1}{2n}\right)^{1/2n}}_{\substack{<1 \\ > \sqrt[2n]{\frac{1}{2}} \to 1}} \cdot \underbrace{(2n)^{1/2n}}_{\to 1}\right)^2 \longrightarrow \dfrac{e^2}{4} = \dfrac{1}{R}$

ⓒ $\sum\limits_{n=1}^{\infty}\left(\prod\limits_{k=1}^{n}\left(1+\dfrac{1}{k}\right)^k\right)x^n$. QK $\dfrac{a_{n+1}}{a_n} = \left(1+\dfrac{1}{n+1}\right)^{n+1} \xrightarrow{n\to\infty} e = \dfrac{1}{R}$

oder WK $\sqrt[n]{a_n} = \left[\left(\dfrac{1+1}{1}\right)^1\left(\dfrac{2+1}{1}\right)^2\cdots\left(\dfrac{n+1}{n}\right)^n\right]^{1/n} = \left(\dfrac{(n+1)^n}{n!}\right)^{1/n} = \dfrac{n+1}{\sqrt[n]{n!}} \to e$

(Abschätzung in Aufg. $\boxed{18}$ $\boxed{\dfrac{1}{e} < \dfrac{\sqrt[n]{n!}}{n+1} < \dfrac{\sqrt[n]{n!}}{n} < \dfrac{1}{e}\sqrt[n]{en}}$)

ⓓ $\sum\limits_{k=0}^{\infty}\left(\dfrac{1}{2}\right)^{k!}x^{3k}$: $\sqrt[k]{\left(\dfrac{1}{2}\right)^{k!}} = \left(\dfrac{1}{2}\right)^{(k-1)!} \le \left(\dfrac{1}{2}\right)^{k-1} \xrightarrow{k\to\infty} 0$

$\Rightarrow R = \infty$, Konvergenz für alle $x \in \mathbb{R}$

26 (1) (a) $\sum a_k$, $\sum b_k$ beide absolut konvergent nach QK:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{3}{k+1} \xrightarrow{k\to\infty} 0, \quad \text{ebenso } \left|\frac{b_{k+1}}{b_k}\right| = \frac{1}{k+1} \to 0$$

$\Rightarrow$ Cauchy-Produkt auch abs. konv.

(b) Leibniz $\Rightarrow$ $\sum a_k$, $\sum b_k$ konvergent. Aber: $\sum_{k=0}^{n} a_{n-k} b_k = \sum_{k=0}^{n} \frac{(-1)^{n-k}(-1)^k}{\sqrt{n-k+1}\sqrt{k+1}} =$

$$= (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{n-k+1}\sqrt{k+1}} \not\to 0, \text{ denn:}$$

$$\sum_{k=0}^{n} \frac{1}{\sqrt{n-k+1}\sqrt{k+1}} \geq \sum_{k=0}^{n} \frac{1}{\sqrt{n+1}} \frac{1}{\sqrt{n+1}} = \frac{n+1}{n+1} = 1$$

$\Rightarrow$ das Cauchy-Produkt $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k} b_k\right)$ divergiert.

[noch zu (1): $\sum_{k=0}^{\infty}\frac{(-3)^k}{k!} \cdot \sum_{j=0}^{\infty}\frac{(-1)^j}{j!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-3)^{n-k}}{(n-k)!}\frac{(-1)^k}{k!}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\sum_{k=0}^{n}\frac{3^{n-k}\, n!}{k!(n-k)!}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}1^k = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(1+3)^n = \sum_{n=0}^{\infty}\frac{(-4)^n}{n!} \qquad (e^{-3}\cdot e^{-1} = e^{-4})$$

]

(2) (a) $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ $(|x|<1)$ $\Rightarrow \frac{1}{(1-x)^2} = \left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 1\cdot x^{n-k}\cdot 1\cdot x^k\right) =$

$$= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} 1 = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\Rightarrow \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)x^k\cdot x^{n-k} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n}(k+1) = \sum_{n=0}^{\infty} x^n\left(\frac{n(n+1)}{2} + n+1\right)$$

$$= \sum_{n=0}^{\infty} x^n \frac{(n+2)(n+1)}{2}$$

$\Rightarrow \sum_{k=0}^{\infty}(k+1)(k+2)x^k = \frac{2}{(1-x)^3}$ für $|x|<1$ (~~geo.~~ geom. Reihe abs. konv $\leadsto$ auch alle Cauchy-Produkte)

oder „rückwärts": $\sum_{0}(k+1)(k+2)x^k =$ (abs. Konv. mit WK prüfen) $= \sum_{n=0}^{\infty} x^n \sum_{j=1}^{n+1} j \cdot 2 = 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)x^k x^{n-k}$

$$= 2\left(\sum_{k=0}^{\infty}(k+1)x^k\right)\cdot\sum_{n=0}^{\infty} x^n = \frac{2}{1-x}\cdot\sum_{k=0}^{\infty}(k+1)x^k = \frac{2}{1-x}\sum_{n=0}^{\infty}\sum_{k=0}^{n} x^k x^{n-k} = \frac{2}{1-x}\left(\sum_{k=0}^{\infty} x^k\right)\sum_{n=0}^{\infty} x^n$$

$$= \frac{2}{(1-x)^3}$$

7.0 | 4

27 $\sin x \cdot \cos y = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{y^{2n-2k}}{(2n-2k)!}$

$$\sin(x+y) = \sum_{n=0}^{\infty} (-1)^n \frac{(x+y)^{2n+1}}{(2n+1)!} \underset{\text{Bin.S.}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \sum_{k=0}^{2n+1} x^k y^{2n+1-k} \cdot \binom{2n+1}{k} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{2n+1} \frac{x^k y^{2n+1-k}}{(2n+1-k)!\, k!} \underset{\sum_{k=0}^{2n+1} a_k = \sum_{k=0}^{n} (a_{2k} + a_{2k+1})}{=} \sum (-1)^n \sum_{k=0}^{n} \left[\frac{x^{2k} y^{2n-2k+1}}{(2n-2k+1)!(2k)!} + \frac{x^{2k+1} y^{2n-2k}}{(2n-2k)!(2k+1)!} \right]$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} \frac{y^{2n-2k+1}}{(2n-2k+1)!} (-1)^{n-k} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{y^{2(n-k)}}{(2(n-k))!} =$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j} \frac{x^{2(n-j)}}{(2n-2j)!} (-1)^j \frac{y^{2j+1}}{(2j+1)!} + \sin x \cdot \cos y = \sin y \cos x + \sin x \cos y$$

■

Ähnlich gilt $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Folgerungen: $\sin 2x = 2 \sin x \cos x$, $\cos 2x = (\cos x)^2 - (\sin x)^2$, $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$,

$$\sin x + \sin y = \sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right) + \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right) = \sin \frac{x+y}{2} \cos \frac{x-y}{2} + \sin \frac{x-y}{2} \cos \frac{x+y}{2} +$$

$$+ \sin \frac{x+y}{2} \cos\left(\frac{-x+y}{2}\right) + \sin\left(\frac{-x+y}{2}\right) \cos \frac{x+y}{2} = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} + 0 \quad \text{etc.} \ldots\ldots$$

28 $a = \underbrace{\sum_{n=0}^{k} z_n g^n}_{z_0} + \sum_{n=1}^{\infty} z_n \frac{1}{g^n} =: z_0, z_1 z_2 z_3 \ldots\ldots$; hier $a = 1{,}18279$, $g = 2$

$z_0 = 1$ — $z_1 = \lfloor (a - z_0) g \rfloor =$ („größtes Ganzes $\leq \ldots$“) $= \lfloor 0{,}36558 \rfloor = 0$ —

$z_2 = \lfloor (a - z_0 - \frac{z_1}{g}) g^2 \rfloor = \lfloor 0{,}73116 \rfloor = 0$

$z_3 = \lfloor (a - z_0 - \frac{z_1}{g} - \frac{z_2}{g^2}) g^3 \rfloor = \lfloor 1{,}46232 \rfloor = 1$

$z_4 = \lfloor \cdots \rfloor = \lfloor 1{,}46232 \cdot 2 - z_3 \cdot 2 \rfloor = \lfloor 0{,}92464 \rfloor = 0$

$z_5 = \lfloor 0{,}92464 \cdot 2 - 0 \cdot 2 \rfloor = \lfloor 1{,}84928 \rfloor = 1$

$z_6 = \lfloor 1{,}84928 \cdot 2 - 1 \cdot 2 \rfloor = \lfloor 1{,}69856 \rfloor = 1$

$z_7 = \lfloor 0{,}69856 \cdot 2 \rfloor = \lfloor 1{,}39712 \rfloor = 1$

$z_8 = \lfloor 0{,}39712 \cdot 2 \rfloor = \lfloor 0{,}79424 \rfloor = 0$ — $z_9 = \lfloor 1{,}58848 \rfloor = 1$

$z_{10} = \lfloor 2 \cdot 0{,}58848 \rfloor = 1$

$\Rightarrow 1{,}18279\ldots = 1{,}0010111011\ldots$

Brechen wir diese Entwicklung nach der 8. Stelle einfach ab, da sehen wir also $\tilde{\alpha} := 1{,}00101110_2$, dann gilt für den Fehler: $|\alpha - \tilde{\alpha}| = 0{,}00000000 11\ldots_2 \geq 2^{-9} + 2^{-10} > 2^{-9}$. Setzen wir dagegen $\hat{\alpha} := 1{,}00101111_2$, dann gilt $|\alpha - \hat{\alpha}| = \hat{\alpha} - \alpha < 2^{-9}$:

$$\alpha - \tilde{\alpha} = \sum_{n=9}^{\infty} g^{-n} z_n = 2^{-9} + 2^{-10} + \sum_{n=11}^{\infty} \cdots \geq 2^{-9} + 2^{-10} > 2^{-9}$$

$$\hat{\alpha} - \alpha = g^{-8} - 0 + \sum_{n=9}^{\infty} g^{-n}(-z_n) = 2^{-8} - 2^{-9} - 2^{-10} - \sum_{11} \cdots \leq \underbrace{2^{-8} - 2^{-9} - 2^{-10}}_{= 2^{-10}}$$

$\Rightarrow$ $\hat{\alpha}$ liegt näher an α als $\tilde{\alpha}$, ist die bessere Näherung auf 8 Nachkommastellen.

16-adische Entw. von $\hat{\alpha} = 1 + 2^{-3} + 2^{-5} + 2^{-6} + 2^{-7} + 2^{-8} = 1 + 2 \cdot 2^{-4} + (8+4+2+1) \cdot 2^{-8} =$

$$= 1 + 2 \cdot 16^{-1} + 15 \cdot 16^{-2} = 1 + 2 \cdot 16^{-1} + F \cdot 16^{-2}$$

$$\Rightarrow \quad \hat{\alpha} = 1{,}2F_{16}$$

(2) $0 \leq \alpha < 1 \Rightarrow \alpha = \sum_{k=1}^{\infty} \hat{z}_k g^{-k}$.

Hier speziell $\alpha = 0{,}z_1 \ldots z_m z_{m+1} \ldots z_n \underbrace{z_{n+1}}_{= z_{m+1}} \ldots z_n \underbrace{z_{n+1}}_{\hat{z}_{m+n-m}} \ldots$,

also
$$\alpha = \sum_{k=1}^{m} z_k g^{-k} + \sum_{k=m+1}^{n} z_k g^{-k} + \sum_{k=m+1}^{n} z_k g^{-(k+n-m)} + \sum_{k=m+1}^{n} z_k g^{-(k+2(n-m))} +$$

$$+ \sum_{k=m+1}^{n} z_k g^{-k} \left(g^{-(n-m)}\right)^3 + \ldots = \underbrace{\sum_{k=1}^{m} z_k g^{-k}}_{=: p \in \mathbb{Q}} + \sum_{j=0}^{\infty} \underbrace{\left(\sum_{k=m+1}^{n} z_k g^{-k}\right)}_{=: q \in \mathbb{Q}} g^{-(n-m)j} =$$

$$= p + \sum_{j=0}^{\infty} \left(g^{-n+m}\right)^j \cdot q = p + \frac{q}{1 - g^{m-n}}, \quad m < n \Rightarrow g^{m-n} < 1$$

$\Rightarrow$ geom. Σ konvergiert

$$\alpha = p + q \frac{g^{n-m}}{g^{n-m} - 1} \in \mathbb{Q}, \text{ da } p, q \in \mathbb{Q},\ g^{n-m} \in \mathbb{N} \setminus \{1\}$$ ■

(3) $\alpha = 0{,}012301230123\ldots_4 < 1$, hier $m = 0$, $n = 4$

$$\Rightarrow \alpha = \sum_{j=0}^{\infty} \left(0 \cdot 4^{-1} + 1 \cdot 4^{-2} + 2 \cdot 4^{-3} + 3 \cdot 4^{-4}\right) 4^{-4 \cdot j} = \sum_{j=0}^{\infty} \frac{1 \cdot 16 + 2 \cdot 4 + 3}{256} \cdot \left(\frac{1}{256}\right)^j =$$

$$= \frac{27}{256} \cdot \frac{1}{1 - \frac{1}{256}} = \frac{27}{255} = \frac{9}{85}$$

$\boxed{27.}$ $\sin x \cdot \cos y = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \frac{(-1)^{n-k}}{(2n-2k)!} x^{2n-2k}$

$= \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \sum_{k=0}^{n} \frac{1}{(2n-2k)!(2k+1)!}$

7.B.6

Ergänzung zu $\boxed{25}$ (2)

(a) $\sum_{k=0}^{\infty} \left(\frac{k!}{3 \cdot 5 \cdot 7 \ldots (2k+1)} \right)^2 x^k$, $R = 4$ (denn 2, da $(\ldots)^1$)

divergent in ~~$x = +4$~~ (bzw. oben $+2$):

$\left(\frac{n!}{3 \ldots (2n+1)} \right)^{\frac{1}{2}} 2^{\frac{1}{2}n} = \left(2^n \frac{1 \cdot 2 \ldots n}{3 \cdot 5 \ldots (2n+1)} \right)^{\frac{1}{2}} > \frac{2^n}{2n+1} \left(\frac{1}{2} \right)^{n-1} > \frac{2}{2n+2}$ div.

mit dem Quadrat: in $+4$?

$x = -2 \mid -4$: $\left(\frac{2^n n!}{3 \ldots (2n+1)} \right)^2$ monoton fallend: $1 \geq \frac{a_{n+1}}{a_n} = \left(\frac{2(n+1)}{2n+3} \right)^2$ ✓

aber gegen 0 ? Falls ja, dann Konv. nach LK.

(c) $\sum_{n=1}^{\infty} x^n \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right)^k$ in $\pm \frac{1}{e}$.

$\frac{a_{n+1}}{a_n} = \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{e} < \frac{e}{e} = 1 \Rightarrow$ fallend. $\frac{1}{e^n} \prod \left(\frac{k+1}{k} \right)^k = \frac{1}{e^n} \left[\frac{2}{1} \left(\frac{3}{2} \right)^2 \left(\frac{4}{3} \right)^3 \cdots \left(\frac{n+1}{n} \right.\right.$

$= \frac{(n+1)^n}{e^n n!} \geq \frac{1}{n+1} \rightsquigarrow$ Div. in $+\frac{1}{e}$

$\frac{1}{e} < \frac{\sqrt[n]{n!}}{n+1} < \frac{\sqrt[n]{n!}}{n} < \frac{1}{e} \sqrt[n]{en} \Rightarrow \frac{n!}{n^n} < \frac{en}{e^n} \Leftrightarrow \frac{n^{n-1}}{(n-1)!\, e^{n-1}} > \frac{1}{n}$

Konv. in $-\frac{1}{e}$ mit Stirling: $\underbrace{\frac{n^n}{n!\, e^n}}_{\to 0} \underbrace{\sqrt{n} \sqrt{2\pi}}_{\to \infty} \to 1$

$\frac{(n+2)^{n+1}}{e^{n+1}(n+1)!} \cdot \frac{e^n n!}{(n+1)^n} \Leftrightarrow \frac{a_{n+1}}{a_n}$ s.o. $\rightsquigarrow$ LK

(b) $-\frac{4}{e^2}$: $\frac{a_{n+1}}{a_n} = \left(\frac{2n+1}{2n-1} \right)^{2n-1} \frac{2n+1}{(2n+2) e^2} < \left(1 + \frac{2}{2n-1}\right)^{\frac{2n-1}{2} \cdot 2} \frac{2n+2}{2n+2} \frac{1}{e^2} < 1$ ✓ $\rightsquigarrow$ fallend

$a_n = \frac{(2n-1)^{2n-1}}{e^{2n-1} (2n-1)!} \frac{1}{e \cdot 2n} \to 0 \cdot 0$ (Stirling, s.o.) $\Rightarrow$ Konv. nach LK

$+\frac{4}{\ }$. ?